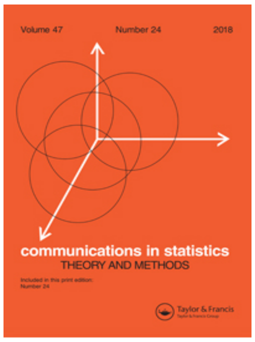


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# Fiducial confidence intervals for proportions in finite populations: One- and two-sample problems

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# Fiducial confidence intervals for proportions in finite populations: One- and two-sample problems

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**ABSTRACT**
The problems of constructing confidence intervals (CIs) for the proportions and functions of proportions in finite populations are considered. For estimating the proportion in a finite population, we propose a CI based on the generalized fiducial method and compare it with an exact CI and score CI. For the two-sample problems, we consider interval estimating the difference between two proportions, the ratio of two proportions and the ratio of odds. Our solutions for the two-sample problems are based on the fiducial approach and the method of variance estimate recovery. All the CIs are evaluated on the basis of their exact coverage probabilities and expected widths. The methods are illustrated using some practical examples.

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## 1. Introduction

A basic yet an important problem in statistical inference is setting confidence interval for a proportion or for some summary indices of proportions. Such problems have been well addressed for proportions in infinite populations on the basis of binomial models. Other important two-sample problems such as the estimation of the difference or the ratio of two proportions assuming binomial models are also well addressed and several approximate solutions are proposed in the literature. See Agresti and Coull (1998), Agresti (1999), Brown, Cai, and Gupta (2001), Krishnamoorthy and Lee (2010), Fagerland and Newcombe (2013), and the references therein. Although there has been continuous interest in developing inferential procedures for binomial distributions, only limited results are available for estimating the proportion in a finite population; see Burstein (1975), Krishnamoorthy and Thomson (2002) and Lee (2009) and Li, Zhou, and Tian (2013). Indeed, to the best of our knowledge, the problems of estimating the relative risk (ratio of proportions) or the ratio of odds from finite populations were never addressed in the literature. Recall that in estimation of a proportion in an infinite population, we deal with a binomial distribution, and in a finite population case we deal with a hypergeometric distribution. Even though these two distributions are practically the same for large populations, results based on binomial distributions are not applicable to hypergeometric distributions when the population sizes are not very large.

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For instance, in the one-sample case, Burstein (1975) noted that the difference between the binomial-based on confidence intervals with finite population correction and the ones based on the hypergeometric distributions are appreciable if the population size is around 5,000 or less.

To describe our present problems formally, consider a hypergeometric distribution with a lot size $N$ and unknown number of defective items $M$. Let $X$ be the number of defective items in a sample of size $n$ drawn from the lot without replacement. For convenience, we write $X \sim H(n, M, N)$. The probability mass function (pmf) of $X$ is given by

$$P(X = x|n, M, N) = f(x|n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \quad L \le x \le U, \tag{1}$$

where $L = \max\{0, M - N + n\}$ and $U = \min\{n, M\}$.

Lee (2009) has considered the problem of estimating the proportion $p = M/N$ of defective items in the lot, and proposed the CIs for $p$ based on the Wald method, Wilson's score method and the Agresti and Coull (1998) method and recommended Wilson's score CI for applications as it is simple to compute and also has good coverage property. Recently, Wang (2015) has derived some exact optimal CIs for proportion in a finite population. The exact CI, like the Clopper and Pearson (1934) exact CI for the binomial proportion, is too conservative, yielding CIs that are unnecessarily wide. Interval estimates for other parameters such as the quantile and tolerance intervals for hypergeometric distributions are proposed in Young (2015). As noted earlier, only very limited results are available for two-sample problems involving hypergeometric distributions. Krishnamoorthy and Thomson (2002) have addressed the problem of testing the equality of two proportions from finite populations and proposed a test that is satisfactory even for small samples. However, the proposed two-sample test in Krishnamoorthy and Thomson (2002) is difficult to invert to find a CI for the difference between two proportions.

In this article, we first address the problem of interval estimating the proportion in a finite population. We then address the problems of interval estimating the difference between two proportions, the ratio of two proportions and the ratio of odds. Keeping these problems in mind, the rest of the article is organized as follows. In the following section, we describe the generalized fiducial method and the method of variance estimate recovery (MOVER) that will be used to find CIs for the aforementioned problems. In Section 3, we consider interval estimating the proportion and describe the score CI, the exact CI by Wang (2015) and a CI based on the generalized fiducial approach by Hannig (2013). In Section 4, we consider two-sample problems where we propose fiducial CIs for the difference between two proportions, the ratio of proportions and for the ratio of odds. Some closed-form approximate CIs are also proposed. In Section 5, the merits of the proposed CIs are evaluated numerically in terms of coverage probabilities and, where applicable, expected widths. All interval estimation methods are illustrated using some examples in Section 6. Some concluding remarks are given in Section 7.

## 2. Preliminaries

We shall now describe a few general methods that will be used in the sequel to find CIs in one- and two-sample problems.

### 2.1. Fiducial distribution for $M_x$

#### 2.1.1. Generalized fiducial distribution

Using the general idea of Hannig (2013), Krishnamoorthy and Lv (2020) obtained a fiducial distribution of $M$ from the hypergeometric random variate generator, which can be described as follows. Recall that $x$ is a pseudo random number from the $H(n, M, N)$ distribution if

$$P(X \leq x-1|n, M, N) < U \leq P(X \leq x|n, M, N), \tag{2}$$

where $U$ is a uniform(0,1) random variable (e.g., see Casella and Berger (2002), page 249). For a given ($x$, $n$, $N$), the support of the fiducial distribution of $M$ is $[x, N-(n-x)]$. A sample from the fiducial distribution of $M$ can be obtained by generating $U_1, ..., U_N$ and then finding the values of $M$ that satisfy the inequality (2) for each $U_i$, $i = , ..., N$. For a given $(x, n, N, U)$, more than one $M$ satisfy the inequality (2). As suggested by Hannig et al. (2016), a randomly selected value from the values of $M$ that satisfy the inequality can be regarded as a fiducial variate. Thus, a sample from the fiducial distribution of $M$ can be obtained as follows. For a generated uniform(0, 1) random number $U_i$, select one element from the set

$$\{M_x : P(X \leq x-1|n, M_x, N) < U_i \leq P(X \leq x|n, M_x, N)\}, \tag{3}$$

at random and refer to the selected element as $M^*_{x,i}$. Then the fiducial sample is given by

$$\{M^*_{x,1}, ..., M^*_{x,N}\}. \tag{4}$$

Fiducial inference can be made on the basis of the fiducial sample. For example, the lower and the upper 5th percentiles of the fiducial sample form a 90% CI for $M$. The following algorithm can be used to generate fiducial samples $M_{x,i}$'s. The R code based on the algorithm is given in the Appendix.

**Algorithm 1**
For a given $x$, $n$ and $N$,

1. Let $S = \{x, x+1, ..., x+N_x-n_x\}$, the support of the fiducial distribution of Mx.
2. Compute the probabilities

   $$P_{0i} = P(X_i \leq x-1|n, S(i), N), \text{ and } P_{1i} = P(X_i \leq x|n, S(i), N), \quad i = 1, ..., N_x - n_x + 1,$$

   where $X_i$ is the hypergeometric random variable with the sample size $n$, number of defective items $S(i)$ and the lot size $N$, and $S(i)$ is the $i$th element of $S$.
3. Generate a $u \sim$ uniform(0,1).
4. Find the set $S^* = \{S(i) : P_{0i} < u \leq P_{1i}, i = 1, ..., N_x - n_x + 1\}$.
5. Select one element at random from $S^*$ and deliver it as a variate of $M_x$.
6. Repeat steps 3–5 until the desired number of fiducial variates generated.

#### 2.1.2. An approximate Z-fiducial distribution

An approximate fiducial distribution for $M_x$ can be found along the lines of Li, Zhou, and Tian (2013) who obtained a *generalized pivotal quantity* for the binomial parameter. The generalized pivotal quantity is the same as the fiducial quantity which can be found on the basis of the functional-model approach by Dawid and Stone (1982). Toward that, we use the asymptotic distributional result (Wald 1943) that, for large $n$,

$$\frac{\hat{p}-p}{\sqrt{\mathrm{Var}(\hat{p})}}=\frac{\hat{p}-p}{\sqrt{Rp(1-p)/n}}\overset{d}{=}Z, \tag{5}$$

where $\hat{p}=(x/n), p=M/N, R=(N-n)/(N-1)$ is the finite population correction, and $Z$ is the standard normal random variable. Solving the above equation for $p$, and then using the fact that $-Z\overset{d}{=}Z$, we find an approximate fiducial distribution for $p$ determined by

$$Q_p=\left(\frac{\hat{p}+\frac{Z^2R}{2n}}{1+\frac{Z^2R}{n}}\right)+\frac{\frac{Z\sqrt{R}}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p})+Z^2R/(4n)}}{1+\frac{Z^2R}{n}}. \tag{6}$$

The above fiducial quantity with $R$ removed is the same as the one given for the binomial parameter in Eqn (3.1) of Li, Zhou, and Tian (2013).

### 2.2. The method of variance estimate recovery

The method of variance estimate recovery (MOVER), introduced by Zou and Donner (2008) and Zou, Taleban, and Huo (2009), is a method to find a CI for a linear combination of parameters based on individual CIs of the parameters. Consider a linear combination $\sum_{i=1}^{k} c_i\theta_i$ of parameters $\theta_1, \ldots., \theta_g$, where $c_i$'s are known constants. Let $\hat{\theta}_i$ be an unbiased estimate of $\theta_i$, $i=1,\ldots,k$. Assume that $\hat{\theta}_1,\ldots,\hat{\theta}_g$ are independent. Furthermore, let $(l_i, u_i)$ denote the $1-\alpha$ confidence interval for $\theta_i$, $i=1,\ldots,k$. The $1-\alpha$ MOVER confidence interval $(L, U)$ for $\sum_{i=1}^{k} c_i\theta_i$ can be expressed as

$$L=\sum_{i=1}^{g} c_i\hat{\theta}_i-\sqrt{\sum_{c_i>0}^{g} c_i^2\left(\hat{\theta}_i-l_i\right)^2+\sum_{c_i<0}^{g} c_i^2\left(\hat{\theta}_i-u_i\right)^2}, \tag{7}$$

and

$$U=\sum_{i=1}^{g} c_i\hat{\theta}_i+\sqrt{\sum_{c_i>0}^{g} c_i^2\left(\hat{\theta}_i-u_i\right)^2+\sum_{c_i<0}^{g} c_i^2\left(\hat{\theta}_i-l_i\right)^2}. \tag{8}$$

#### 2.2.1. MOVER CI for the ratio of two parameters

To describe the MOVER CI for the ratio $\eta=\theta_1/\theta_2$ of two parameters, where both parameters $\theta_1$ and $\theta_2$ are positive, let $\hat{\theta}_i$ be an estimator of $\theta_i$, $i=1, 2$. Assume that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent. Furthermore, let $(l_i, u_i)$ denote the $100(1-\alpha)\%$ confidence interval for $\theta_i$, $i=1, 2$. The MOVER confidence limits for the ratio can be obtained from the MOVER confidence limits for $\theta_1 - \eta\theta_2$ using Fieller's (1940) theorem. To be more specific, let $L^*(\eta, \hat{\theta}_1, \hat{\theta}_2)$ denote the left endpoint of the $1-\alpha$ MOVER CI for $\theta_1 - \eta\theta_2$ based on Equation (7). Then the lower limit of MOVER CI for the ratio $\theta_1/\theta_2$ is the root (with respect to $\eta$) of the equation $L^*(\eta; \hat{\theta}_1, \hat{\theta}_2) = 0$. The left endpoint of the CI for the ratio is given by

$$L_\eta = \frac{\hat{\theta}_1\hat{\theta}_2 - \sqrt{\hat{\theta}_1^2\hat{\theta}_2^2 - \left[\hat{\theta}_2^2 - (u_2 - \hat{\theta}_2)^2\right]\left[\hat{\theta}_1^2 - (l_1 - \hat{\theta}_1)^2\right]}}{\hat{\theta}_2^2 - (u_2 - \hat{\theta}_2)^2}. \quad (9)$$

Similarly, if $U^*(\eta, \hat{\theta}_1, \hat{\theta}_2)$ denote the right endpoint of the $1-\alpha$ MOVER CI for $\theta_1 - \eta\theta_2$ based on Equation (8), then the upper limit of MOVER CI for the ratio $\theta_1/\theta_2$ is the root of the equation $U^*(\eta; \hat{\theta}_1, \hat{\theta}_2) = 0,$ and is given by

$$U_\eta = \frac{\hat{\theta}_1\hat{\theta}_2 + \sqrt{\hat{\theta}_1^2\hat{\theta}_2^2 - \left[\hat{\theta}_1^2 - (u_1 - \hat{\theta}_1)^2\right]\left[\hat{\theta}_2^2 - (l_2 - \hat{\theta}_2)^2\right]}}{\hat{\theta}_2^2 - (\hat{\theta}_2 - l_2)^2}. \quad (10)$$

Donner and Zou (2012) first derived the above CI from the one for $\theta_1 - \eta\theta_2$. Although the above expressions for $L_\eta$ and $U\eta$ are different from those given in Donner and Zou (2012), it can be verified that they are the same. Krishnamoorthy, Peng, and Zhang (2016) have obtained the same CI for the ratio of Poisson means by inverting the one-sided tests for $\theta_1 - \eta\theta_2$.

## 3. Confidence intervals for *M*

For a given $x$, $n$ and $N$, we shall describe the score CI, an exact CI and fiducial CIs in the sequel.

### 3.1. Score confidence interval

Score CI for $M_x$ is based on the approximate distributional result in Equation (5). Letting $q_\alpha = z_{1-\alpha/2}$ to denote the $1-\alpha/2$ quantile of the standard normal distribution, the $1-\alpha$ score CI for $p$ can be expressed as

$$(\hat{p}_L, \hat{p}_U) = \left(\frac{\hat{p} + \frac{q_\alpha^2 R}{2n}}{1 + \frac{q_\alpha^2 R}{n}}\right) \mp \frac{\frac{q_\alpha\sqrt{R}}{\sqrt{n}}\sqrt{\hat{p}(1-\hat{p}) + q_\alpha^2 R/(4n)}}{1 + \frac{q_\alpha^2 R}{n}}, \quad (11)$$

The CI for $M$ on the basis of the above CI is given by

$$[M_L, M_U] = \left[\lceil N\hat{p}_L \rceil, \lfloor N\hat{p}_U \rfloor\right], \quad (12)$$

where $\lceil x \rceil$ is the ceiling function and $\lfloor x \rfloor$ is the floor function.

**Remark.** It should be noted that the above score CI is the same as the one based on the approximate $Z$-fiducial quantity $Q_p$ in Equation (6). Specifically, let $Q_{p;\alpha}$ denote the $100\alpha$ percentile of $Q_p$ when $x$ is fixed. Then the $1-\alpha$ fiducial CI is $(Q_{p;\alpha/2}, Q_{p;1-\alpha/2})$ which is the same the one in Equation (11).

### 3.2. Exact confidence intervals for $M$

Let $x$ be an observed value of $X \sim H(n, M, N)$, and consider testing $H_0 : M = M_0$ vs. $H_a : M > M_0$ at a level of significance $\alpha$. The test that rejects the null hypothesis if the p-value $P(X \geq x|n, M_0, N) \leq \alpha$ is a uniformly most powerful (UMP) test, because the family of hypergeometric distributions has monotone likelihood ratio property. So the one-sided confidence interval that is obtained by inverting this UMP test is uniformly most accurate (see Section 9.3.2 of Casella and Berger (2002)). Let $M_L$ be the smallest integer such that

$$P(X \geq x|n, M_L, N) \geq \alpha.$$

For any $M_0 \geq M_L$, the p-value for testing the right-sided hypothesis is greater than $\alpha$, and so the $M_L$ is a $100(1-\alpha)\%$ lower confidence limit for $M$. Similarly, by inverting the test for $H_0 : M = M_0$ vs. $H_a : M < M_0$, it can be seen that an upper confidence limit $M_U$ for $M$ is the largest integer such that

$$P(X \leq x|n, M_U, N) \geq \alpha.$$

Wang (2015) has shown that, for any $1-\alpha$ lower confidence limit $M_L^*$ with nondecreasing in $x$, $M_L \geq M_L^*$. Similarly, for any $M_U^*$ with nondecreasing in $x$, $M_U \leq M_U^*$.

Even though the one-sided confidence limits are uniformly most accurate (UMA), the $1-2\alpha$ two-sided confidence interval $[M_L, M_U]$, formed by these one-sided limits, is not UMA. Wang (2015) has proposed an iterative algorithm to find an exact admissible two-sided CI $[M_{eL}, M_{eU}]$ from $[M_L, M_U]$. Calculation of the admissible two-sided CI is numerically quite involved; the R program provided by Wang is a function of even $n$, $N$ and $1-\alpha$, and returns $(n+1)$ CIs for $x \in \{0, 1, ..., n\}$. In the sequel, we shall refer to Wang's admissible two-sided CI as the exact CI.

### 3.3. Fiducial CI for $M$

Given $x$, $n$ and $N$, the fiducial CI is obtained by the percentiles of the sample in Equation (4) generated from the fiducial distribution of $M$. Specifically, let $[L_f, U_f]$ denote the [lower, upper] $100\alpha$ percentile of the sample in Equation (4). Then, $[L_f, U_f]$ is a $1-2\alpha$ fiducial CI for $M$.

## 4. Confidence intervals for two-sample problems

Let $X \sim H(n_x, M_x, N_x)$ independently of $Y \sim H(n_y, M_y, N_y)$. Let $p_x = M_x/N_x$ and $p_y = M_y/N_y$. We shall now see a few methods of finding CIs for the difference between proportions, ratio of proportions and for the ratio of odds.

### 4.1. Confidence intervals for the difference $p_x - p_y$

#### 4.1.1. Fiducial CI

A fiducial CI for the ratio can be obtained on the basis of independent fiducial samples. For a given $(x, n_x, N_x)$, let $M_{x,1}^*, ..., M_{x,N}^*$ denote the fiducial sample, which can be generated using the R-code in the Appendix. Similarly, find the fiducial sample

$M^*_{y,1}, \ldots, M^*_{y,N}$ for a given $(y, n_y, N_y)$. Let

$$D_i = (M^*_{x,i}/N_x) - (M^*_{y,i}/N_y), \; i = 1, \ldots, N.$$

The lower and upper $100\alpha$ percentiles of the $D_i$'s is a $1-2\alpha$ fiducial CI for $p_x - p_y$.

#### 4.1.2. Z-fiducial CI

Let $\hat{p}_x = (x/n), p_x = M_x/N_x, R_x = (N_x - n_x)/(N_x - 1)$ and let $Z_x$ denote the standard normal random variable. Following Equation (6), define the fiducial variable

$$Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x) = \left( \frac{\hat{p}_x + \frac{Z_x^2 R_x}{2n}}{1 + \frac{Z_x^2 R_x}{n_x}} \right) + \frac{\frac{Z_x \sqrt{R_x}}{\sqrt{n_x}} \sqrt{\hat{p}_x(1-\hat{p}_x) + Z_x^2 R_x/(4n_x)}}{1 + \frac{Z_x^2 R_x}{n_x}}. \quad (13)$$

Similarly, define $Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y)$ on the basis of $(y, n_y, N_y)$. Then the fiducial quantity for the difference $p_x - p_y$ is given by

$$D^* = Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x) - Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y). \quad (14)$$

The lower and upper $100\alpha$ percentiles of the $D^*$'s is a $1-2\alpha$ fiducial CI for $p_x - p_y$. For a given $x$ and $y$, the percentiles of $D^*$ can be estimated by generating independent standard normal random variables $Z_x$ and $Z_y$.

#### 4.1.3. A closed-form approximate fiducial CI

A closed-form approximate fiducial CI can be obtained by approximating the percentiles of $D^*$ in Equation (14). To find an approximation to a percentile of $D^*$, we shall use the modified normal-based approximation described in Krishnamoorthy (2016). Notice that, for a given $(x, n_x, N_x, y, n_y, N_y)$, $Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x)$ and $Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y)$ are independent with the

$$\text{med}\big(Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x)\big) = Q_{p_x}(z_{.5}; \hat{p}_x, n_x, R_x) = \hat{p}_x,$$

where $z_{.5} = 0$ is the median of the standard normal distribution. Similarly, we see that the median of $Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y)$ is $\hat{p}_y$. Let

$$l_x = Q_{p_x}(z_\alpha; \hat{p}_x, n_x, R_x) \quad \text{and} \quad u_x = Q_{p_x}(z_{1-\alpha}; \hat{p}_x, n_x, R_x),$$

where $z_\alpha$ denote the $\alpha$ quantile of the standard normal distribution. Similarly, define $l_y = Q_{p_y}(z_\alpha; \hat{p}_y, n_y, R_y)$ and $u_y = Q_{p_y}(z_{1-\alpha}; \hat{p}_y, n_y, R_y)$. Then an approximate $100\alpha$ percentile of $D^*$ is given by

$$L = \hat{p}_x - \hat{p}_y - \sqrt{\left(\hat{p}_x - l_x\right)^2 + \left(\hat{p}_y - u_y\right)^2}, \quad (15)$$

and an approximate $100\,(1-\alpha)$ percentile is given by

$$U = \hat{p}_x - \hat{p}_y + \sqrt{\left(\hat{p}_x - u_x\right)^2 + \left(\hat{p}_y - l_y\right)^2}. \quad (16)$$

The interval $(L, U)$ is an approximate $1-2\alpha$ CI for the difference $p_x - p_y$.

## 4.2. Confidence intervals for the ratio $p_x/p_y$

### 4.2.1. Fiducial CI

A fiducial CI for the ratio can be obtained along the lines for the one for the difference. Let $M^*_{x,1}, \ldots, M^*_{x,N}$ denote the fiducial sample based on $(x, n_x, N_x)$, let $M^*_{y,1}, \ldots, M^*_{y,N}$ denote the fiducial sample based on $(y, n_y, N_y)$. Let

$$R_i = \frac{M^*_{x,i}/N_x}{M^*_{y,i}/N_y}, i = 1, \ldots, N.$$

The lower and upper 100α percentiles of the $R_i$'s is $1-2\alpha$ fiducial CI for $p_x/p_y$.

### 4.2.2. Z-fiducial CI

For a given $(x, n_x, N_x)$, define the fiducial variable $Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x)$ as in Equation (13). Similarly, define $Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y)$ on the basis of $(y, n_y, N_y)$. Then a fiducial quantity for the ratio $p_x/p_y$ is given by

$$R^* = \frac{Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x)}{Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y)} \tag{17}$$

The lower and upper 100α percentiles of the $R^*$ is a $1-2\alpha$ fiducial CI for $p_x/p_y$.

### 4.2.3. MOVER confidence interval

Let us describe the MOVER CI for the ratio based on the individual exact CIs for $p_x$ and $p_y$ that can be obtained using Wang's (2015) algorithm. Let $(l_x,\ u_x)$ and $(l_y,\ u_y)$ denote the $1-\alpha$ exact CIs for $p_x$ and $p_y$, respectively. Furthermore, let $\tilde{p}_x$ and $\tilde{p}_y$ denote the centers of the CIs $(l_x,\ u_x)$ and $(l_y,\ u_y)$, respectively. Following Equations (9) and (10), the MOVER CI $(L,\ U)$ for the ratio can be expressed as

$$L = \frac{\tilde{p}_x\tilde{p}_y - \sqrt{(\tilde{p}_x\tilde{p}_y)^2 - \left[\tilde{p}_y^2 - (u_y - \tilde{p}_y)^2\right]\left[\tilde{p}_x^2 - (l_x - \tilde{p}_x)^2\right]}}{\tilde{p}_y^2 - (u_y - \tilde{p}_y)^2},$$

and

$$U = \frac{\tilde{p}_x\tilde{p}_y + \sqrt{\tilde{p}_y^2\tilde{p}_x^2 - \left[\tilde{p}_x^2 - (\tilde{p}_x - u_x)^2\right]\left[\tilde{p}_y^2 - (\tilde{p}_y - l_y)^2\right]}}{\tilde{p}_y^2 - (\tilde{p}_y - l_y)^2}.$$

## 4.3. Confidence intervals for the odds ratio

### 4.3.1. Fiducial CI

Let $M^*_{x,1}, \ldots, M^*_{x,N}$ and $M^*_{y,1}, \ldots, M^*_{y,N}$ be fiducial samples as defined in Section 4.1. Define

$$O_i = \frac{\left[M^*_{x,i}/(N_x - M^*_{x,i})\right]}{\left[M^*_{y,i}/(N_y - M^*_{y,i})\right]}, i = 1, ..., N.$$

The lower and upper $100\alpha$ percentiles of the $O_i$'s is $1-2\alpha$ fiducial CI for the ratio of odds $[p_x/(1-p_x)]/[p_y/(1-p_y)]$.

#### 4.3.2. Z-fiducial CI

The $Z$-fiducial CI is obtained using the $Z$-fiducial quantities defined in the preceding section. Specifically, the $Z$-fiducial variable for the odds ratio is obtained by substitution as

$$O_{xy} = \frac{Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x)/\left[1 - Q_{p_x}(Z_x; \hat{p}_x, n_x, R_x)\right]}{Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y)/\left[1 - Q_{p_y}(Z_y; \hat{p}_y, n_y, R_y)\right]}.$$

The lower and upper $100\alpha$ percentile of $O_{xy}$ form a $1-2\alpha$ CI for the ratio of odds, and the percentiles can be estimated using Monte Carlo simulation.

#### 4.3.3. The MOVER CI

To construct a MOVER CI for the odds ratio, we first find CIs for the odds $p_x/(1-p_x)$ and $p_y/(1-p_y)$. Let $(l_x, u_x)$ and $(l_y, u_y)$ denote the $1-\alpha$ exact CIs for $p_x$ and $p_y$, respectively. On the basis of the exact CIs, we see that $(l_{ox}, u_{ox}) = (l_x/(1-l_x), u_x/(1-u_x))$ is a $1-\alpha$ CI for odds $p_x/(1-p_x)$ and $(l_{oy}, u_{oy}) = (l_y/(1-l_y), u_y/(1-u_y))$ is a $1-\alpha$ CI for the odds $p_y/(1-p_y)$. Let $\hat{O}_x$ and $\hat{O}_y$ denote the midpoints of the CIs $(l_{ox}, u_{ox})$ and $(l_{oy}, u_{oy})$, respectively. Substituting $(l_{ox}, u_{ox})$ for $(l_1, u_1)$, $(l_{oy}, u_{oy})$ for $(l_2, u_2)$, $\hat{\theta}_1 = \hat{O}_x$ and $\hat{\theta}_2 = \hat{O}_y$ in Equations (9) and (10), we can find a MOVER CI for the ratio of odds.

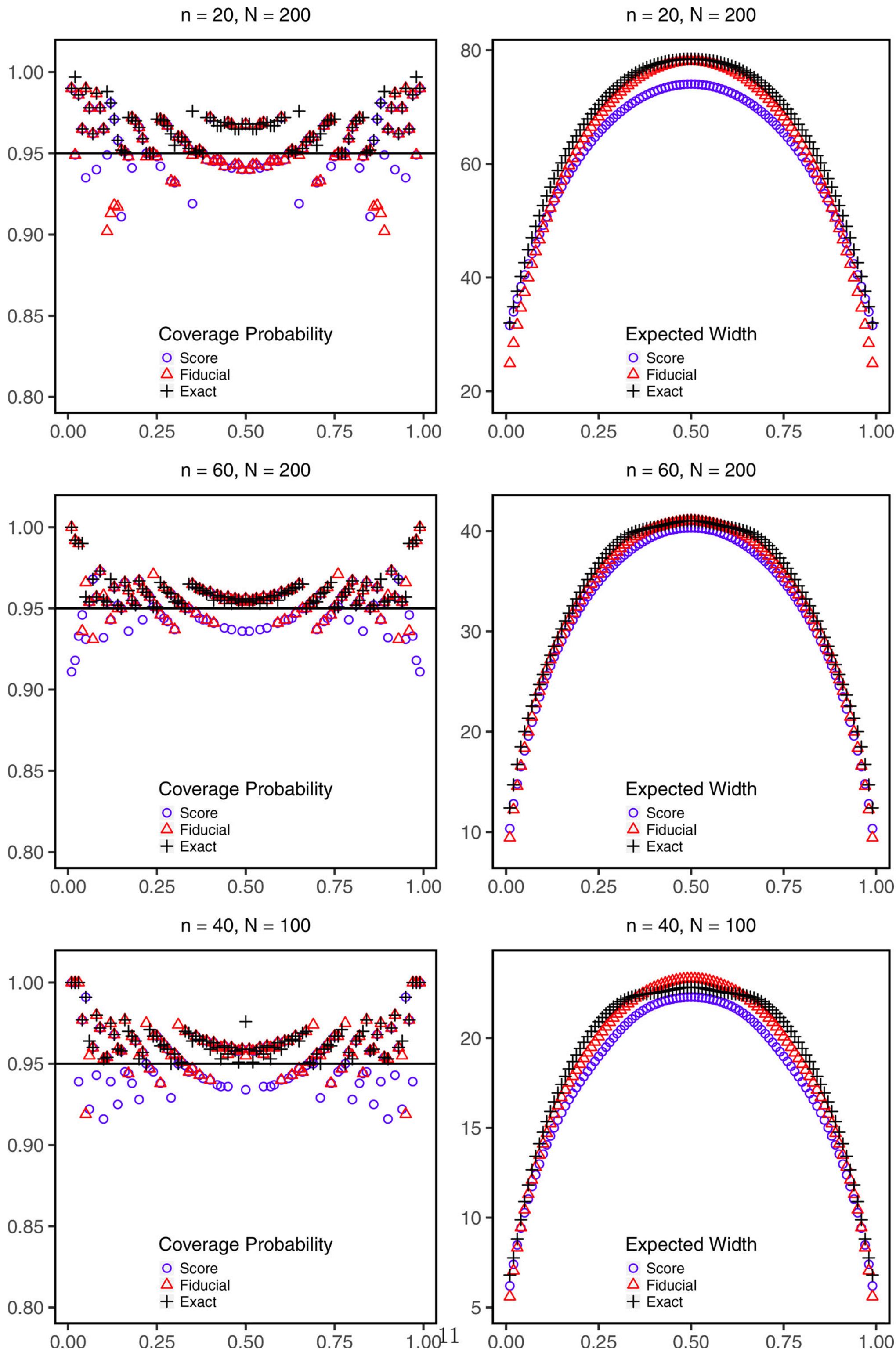
## 5. Coverage and expected widths of CIs for $M$

Let $[l(x, n, N; \alpha), u(x, n, N; \alpha)]$ be a $1-\alpha$ CI for $M$. For an assumed value of $(n, M, N)$, the exact coverage probabilities of $[l(x, n, N; \alpha), u(x, n, N; \alpha)]$ can be evaluated using the hypergeometric pmf $f(x|n, M, N)$ as

$$\sum_{x=L}^{U} f(x|n, M, N) I_{[l(x,n,N;\alpha), u(x,n,N;\alpha)]}(M), \tag{18}$$

where $L$ and $U$ are as defined in Equation (1) and $I_A(x)$ is the indicator function. For a satisfactory CI, the coverage probabilities should be close to the nominal level $1-\alpha$ for all values of $(n, M, N)$. The expected width of a CI is evaluated using the above expression with the indicator function replaced by the width $u(x, n, N; \alpha) - l(x, n, N; \alpha)$.

The coverage probabilities along with expected widths are plotted as a function of $p = M/N$ in Figure 1 for various values of $(n, M, N)$. We observe from the six plots of the coverage probabilities that the exact CI is overly conservative for all the cases considered. The score CI and the fiducial CI are liberal for some parameter values, but




11

**Figure 1.** Coverage probabilities and expected widths 95% confidence intervals of $M_x$ as functions of proportion of defective; $x-$ axis = proportion $M/N$, $y-$ axis = coverage probability.

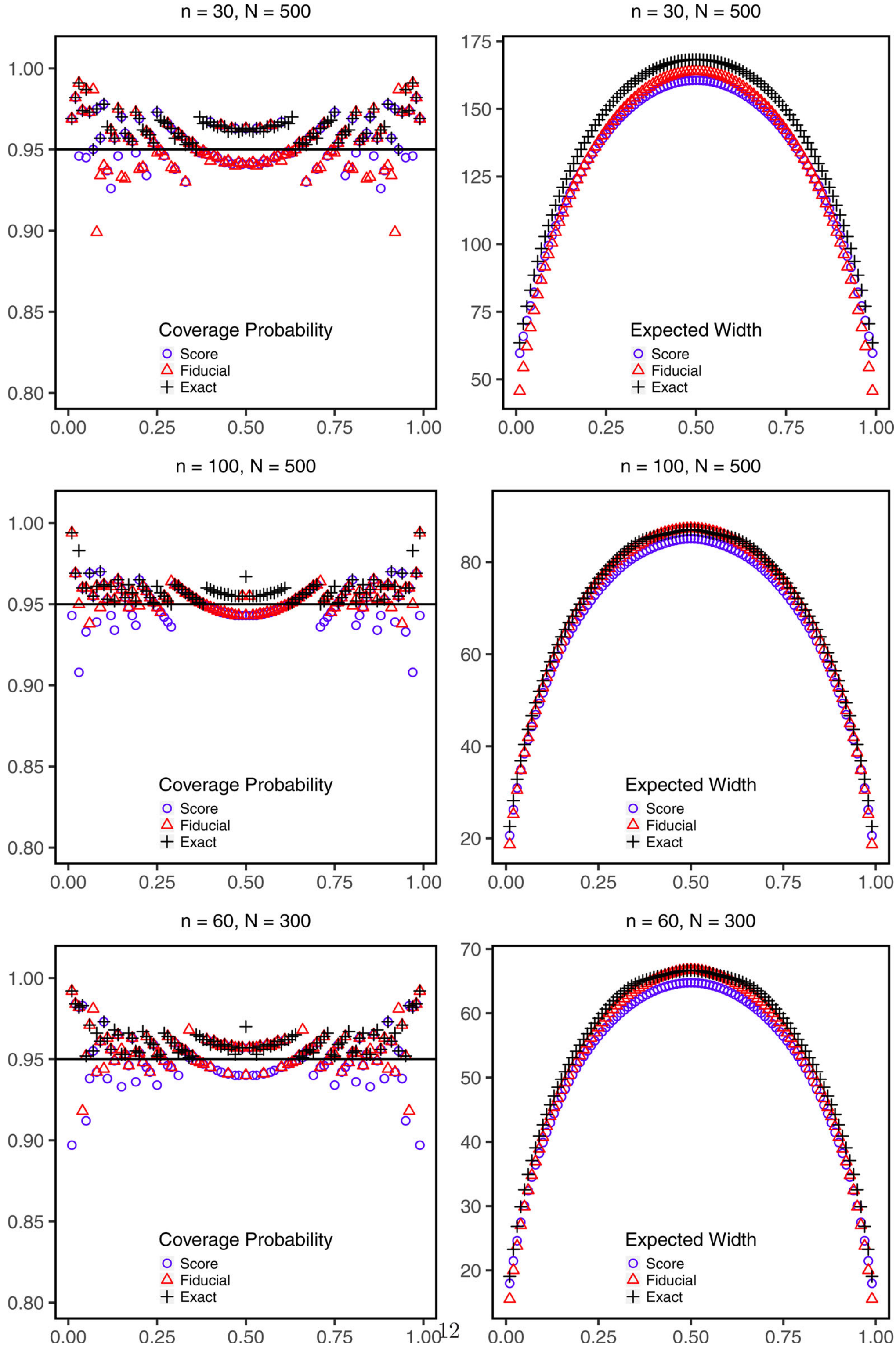





**Figure 1.** (Continued)

**Table 1.** Coverage probabilities of 95% CIs for the difference between proportions.

| Cov Prob | $N_x = 200, N_y = 200$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $(n_x, n_y) = (14, 12)$ | | | $(n_x, n_y) = (20, 24)$ | | | $(n_x, n_y) = (30, 40)$ | | |
| | Fiducial | Z-Fid | A-Fid | Fiducial | Z-Fid | A-Fid | Fiducial | Z-Fid | A-Fid |
| min | .897 | .895 | .897 | .895 | .895 | .906 | .915 | .924 | .924 |
| 5th | .934 | .935 | .934 | .938 | .940 | .939 | .942 | .945 | .944 |
| med | .948 | .950 | .951 | .949 | .950 | .950 | .949 | .950 | .950 |
| 95th | .966 | .978 | .981 | .961 | .967 | .973 | .957 | .963 | .966 |
| | $N_x = 400, N_y = 500$ | | | | | | | | |
| | $(n_x, n_y) = (14, 12)$ | | | $(n_x, n_y) = (20, 24)$ | | | $(n_x, n_y) = (30, 40)$ | | |
| min | .896 | .905 | .905 | .913 | .924 | .913 | .912 | .901 | .882 |
| 5th | .932 | .935 | .932 | .939 | .940 | .939 | .942 | .945 | .945 |
| med | .948 | .952 | .952 | .949 | .950 | .950 | .949 | .950 | .950 |
| 95th | .966 | .979 | .984 | .960 | .967 | .971 | .958 | .965 | .967 |
| | $N_x = 1000, N_y = 1100$ | | | | | | | | |
| | $(n_x, n_y) = (14, 12)$ | | | $(n_x, n_y) = (20, 24)$ | | | $(n_x, n_y) = (30, 40)$ | | |
| min | .899 | .917 | .904 | .907 | .922 | .905 | .933 | .940 | .924 |
| 5th | .932 | .936 | .933 | .939 | .941 | .939 | .943 | .945 | .944 |
| med | .948 | .952 | .952 | .948 | .951 | .951 | .950 | .950 | .949 |
| 95th | .964 | .979 | .984 | .959 | .971 | .976 | .957 | .963 | .967 |

their coverage probabilities are seldom as low as 0.90 when the confidence coefficient is 0.95. These two CIs perform very similar in most cases; however, when the sample sizes are large in relation to the values of $N$, the coverage probabilities of the fiducial CI are more close to the nominal level than those of the score CI. See the plots for $(n, N) = (100, 500)$ and $(60, 300)$. The expected widths of different CIs reflect the coverage properties. Specifically, the exact CI is too conservative and so its expected width is larger than those of other two CIs in most cases. The score CI is shorter than other two CIs when $p$ is in the middle of the interval (0, 1). This higher precision of the score CI could be due to the fact that the coverage probabilities are slightly smaller than the nominal level when $p$ is around 0.5.

### 5.1. Coverage probabilities for two-sample problems

To judge the interval estimates of the difference between two proportions, we evaluated the exact coverage probabilities using the expression

$$\sum_{x=L_x}^{U_x} \sum_{x=L_y}^{U_y} f(x|n_x, M_x, N_x) f(y|n_y, M_y, N_y) I_{(L_d, U_d)}(p_x - p_y), \tag{19}$$

where $(L_d, U_d)$ is a CI for the difference $p_x - p_y$.

Summary statistics of coverage probabilities and expected widths of the fiducial CI, $Z$-fiducial CI and approximate fiducial CI for the difference $p_x - p_y$ are reported in Table 1 for some assumed values of $(n_x, N_x, n_y, N_y)$. For each set of assumed values of $(n_x, N_x, n_y, N_y)$, the summary statistics are based on 1000 coverage probabilities calculated at 1000 pairs of $(p_x, p_y)$ generated randomly from uniform $(.001, .999)$ distribution. Examination of summary statistics in Table 1 clearly indicates that all three CIs

**Table 2.** Percentiles of overage probabilities of 95% CIs for the ratio of proportions.

| | $N_x = 200, N_y = 200$ | | | | | | | | | $N_x = 150, N_y = 500$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $(n_x, n_y)$ | | | | | | | | | | | | | | | | | |
| Cov Prob | (14,12) | | | (20,24) | | | (30,40) | | | (14,12) | | | (20,24) | | | (30,40) | | |
| | Fid | Z-fid | MOV | Fid | Z-fid | MOV | Fid | Z-fid | MOV | Fid | Z-fid | MOV | Fid | Z-fid | MOV | Fid | Z-fid | MOV |
| min | .889 | .855 | .913 | .900 | .875 | .880 | .879 | .882 | .870 | .888 | .858 | .903 | .867 | .867 | .867 | .801 | .800 | .800 |
| 5th | .928 | .939 | .949 | .934 | .943 | .945 | .933 | .942 | .946 | .924 | .939 | .948 | .935 | .943 | .950 | .935 | .943 | .947 |
| med | .949 | .953 | .969 | .949 | .951 | .964 | .949 | .948 | .959 | .947 | .952 | .968 | .949 | .952 | .965 | .949 | .949 | .960 |
| 95th | .977 | .974 | .995 | .976 | .971 | .986 | .972 | .967 | .977 | .975 | .973 | .995 | .976 | .969 | .985 | .970 | .968 | .979 |
| | $N_x = 400, N_y = 500$ | | | | | | | | | $N_x = 1000, N_y = 1100$ | | | | | | | | |
| | $(n_x, n_y)$ | | | | | | | | | | | | | | | | | |
| Cov Prob | (14,12) | | | (20,24) | | | (30,40) | | | (14,12) | | | (20,24) | | | (30,40) | | |
| min | .906 | .893 | .924 | .904 | .891 | .902 | .903 | .886 | .931 | .889 | .888 | .929 | .905 | .890 | .924 | .909 | .931 | .899 |
| 5th | .932 | .940 | .952 | .939 | .942 | .952 | .941 | .944 | .949 | .928 | .937 | .952 | .939 | .943 | .951 | .940 | .951 | .945 |
| med | .948 | .953 | .971 | .950 | .952 | .966 | .950 | .950 | .962 | .948 | .952 | .971 | .955 | .952 | .966 | .950 | .963 | .951 |
| 95th | .974 | .974 | .994 | .975 | .970 | .987 | .960 | .965 | .980 | .975 | .973 | .993 | .976 | .969 | .987 | .972 | .980 | .966 |

are quite comparable in terms of coverage probabilities. The minimum coverage probabilities of all three methods are close to 0.90 and the median is close to 0.95 when the nominal level is 0.95. Comparison of the median and 95th percentiles of coverage probabilities indicate that the $Z$-fiducial CI and the approximate fiducial CI are slightly more conservative than the fiducial CI. As the expected widths of all three CIs are very similar they are not reported in Table 1. On the basis of simplicity, the Z-fiducial and the approximate fiducial CIs may be recommended for practical use.

The summary statistics of coverage probabilities of fiducial CI, Z-fiducial CI and the MOVER CI for the ratio of proportions are reported in Table 2. The fifth percentiles of the coverage probabilities indicate that all CIs are satisfactory, and they rarely undercover the true ratio. The MOVER CI is mostly conservative, guaranteeing coverage probability for most cases. The fiducial and Z-fiducial CIs perform very similarly in most cases. However, it should be noted that the Z-fiducial CI is easy to calculate compared to the fiducial CI. The MOVER CI appears to be more conservative than the other two CIs.

The results on coverage probabilities of CIs for the odds ratio are reported in Table 3. The summary statistics of the coverage probabilities were calculated as in the preceding paragraphs. At first we see that the MOVER CI is conservative in most cases, even though its coverage probability falls below the nominal level in some cases. This MOVER CI could be overly conservative for small sample sizes. Comparison of the coverage probabilities of the fiducial and $Z$-fiducial CIs indicate that the later one is slightly more conservative than the former one; otherwise, these CIs perform similar with respect to coverage probability.

Overall, we see that the Z-fiducial approach is conceptually simple and produces satisfactory results for interval estimating the difference, ratio and the ratio of odds.

## 6. Examples

**Example 1.** To illustrate the different methods of interval estimation in the preceding sections, we shall adapt the example in Krishnamoorthy and Thomson (2002). This

**Table 3.** Percentiles of coverage probabilities of 95% CIs for the odds ratio.

| | $N_x = 200, N_y = 200$ | | | | | | | | | $N_x = 150, N_y = 500$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Cov Prob | $(n_x, n_y)$ (14,12) | | | (20,24) | | | (30,40) | | | (14,12) | | | (20,24) | | | (30,40) | | |
| | Fid | Z-Fid | MOV | Fid | Z-Fid | MOV | Fid | Z-Fid | MOV | Fid | Z-Fid | MOV | Fid | Z-Fid | MOV | Fid | Z-Fid | MOV |
| min | .912 | .914 | .964 | .894 | .891 | .913 | .876 | .853 | .869 | .907 | .912 | .961 | .908 | .888 | .920 | .837 | .822 | .828 |
| 5th | .930 | .937 | .971 | .932 | .941 | .963 | .934 | .943 | .956 | .929 | .939 | .967 | .933 | .943 | .965 | .934 | .942 | .958 |
| med | .947 | .960 | .986 | .946 | .955 | .978 | .948 | .951 | .966 | .948 | .961 | .986 | .947 | .954 | .977 | .947 | .949 | .967 |
| 95th | .982 | .980 | .998 | .977 | .976 | .992 | .975 | .974 | .986 | .982 | .981 | .997 | .977 | .976 | .991 | .974 | .971 | .988 |
| | $N_x = 400, N_y = 500$ | | | | | | | | | $N_x = 1000, N_y = 1100$ | | | | | | | | |
| | $(n_x, n_y)$ | | | | | | | | | | | | | | | | | |
| Cov Prob | (14,12) | | | (20,24) | | | (30,40) | | | (14,12) | | | (20,24) | | | (30,40) | | |
| min | .910 | .917 | .955 | .920 | .911 | .946 | .911 | .871 | .930 | .913 | .923 | .966 | .919 | .917 | .959 | .923 | .920 | .955 |
| 5th | .932 | .937 | .965 | .934 | .945 | .954 | .936 | .946 | .938 | .932 | .940 | .972 | .934 | .941 | .963 | .937 | .940 | .958 |
| med | .947 | .960 | .985 | .947 | .958 | .971 | .947 | .952 | .953 | .947 | .960 | .987 | .948 | .954 | .977 | .948 | .947 | .967 |
| 95th | .980 | .981 | .998 | .977 | .978 | .995 | .975 | .971 | .994 | .981 | .982 | .998 | .978 | .975 | .994 | .976 | .968 | .990 |

example involves the problem of estimating the the number of unacceptable cans produced by a canning machine. A can is determined to be unacceptable (for sale) if the content of the can weighs less than 95% of the labeled weight. Inspection of a sample of $n_x = 20$ cans from a lot of $N_x = 200$ cans revealed $x = 2$ unacceptable cans.

The computed 95% CIs based on various methods are as follows: Score CI is [6, 57]; fiducial CI based on a fiducial sample of size 10,000 is [5, 55]; the exact CI is [4, 61]. As the exact CI is, in general, conservative, it is wider than the other two CIs.

**Example 2.** This example is also taken from Krishnamoorthy and Thomson (2002) which involves comparing proportions of nonacceptable cans produced by two different canning machines. Denote $p_1$ as the proportion of nonacceptable cans for the first machine and $p_2$ as the proportion for the second machine. Two pallets produced from each machine, each containing $N_1 = N_2 = 250$ cans, are arbitrarily selected. Inspection of sample of 110 cans from machine 1 revealed 8 nonacceptable cans, and a sample of 110 cans from machine 2 revealed 3 nonacceptable cans. Thus, we have we $N_1 = N_2 = 250, n_1 = n_2 = 110, x_1 = 8$ and $x_2 = 3$.

We calculated 95% CIs for $p_1 - p_2$ based on various methods as follows: The fiducial CI is (.004, .092), the Z-fiducial CI is (.002, .093) and the approximate fiducial CI is (.001, .093). All three CIs indicate that the proportion of unacceptable cans produced by machine 1 is greater than that of unacceptable cans produced by machine 2. The 95% CIs for the ratio $p_1/p_2$ are as follows: The fiducial CI is (1.06, 7.00), the Z-fiducial CI is (1.03, 6.93) and the MOVER CI based on the exact CIs for $p_1$ and $p_2$ is (.95, 6.96). Notice that the fiducial and Z-fiducial CIs indicate that $p_1 > p_2$ while the MOVER CI, being conservative in most cases, does not indicate $p_1 > p_2$. The 95% CIs for the ratio of odds are as follows: The fiducial CI is (1.05, 7.52), the Z-fiducial CI is (1.04, 7.54), and the MOVER CI on the basis of individual exact CIs for $p_1$ and $p_2$ is (.95, 7.64). We once again see that both fiducial CIs indicate that odds are significantly different while the MOVER CI indicates they are not.

## 7. Conclusions

In 1930s, Fisher (1930, 1935) introduced the concept of fiducial inference and described a method of obtaining fiducial distributions for parameters by inverting hypothesis tests. In general, fiducial distribution for a parameter is not unique and a few different methods are available to find a fiducial distribution. In this article, we used Hannig's (2013) generalized fiducial approach to find a fiducial distribution for $M$, the number of defective items in a hypergeometric distribution. We also obtained another fiducial distribution for $M/N$ on the basis of the approximate distribution of $Z$-score statistic. The proposed method of generating fiducial samples on $M$ or the $Z$-fiducial samples is conceptually simple and is easy to use to find CIs for various problem involving proportions in finite populations. We showed that the generalized fiducial approach and the $Z$ fiducial approach produced results that are comparable to or better than the results based on other methods for some problems. Our extensive coverage studies for all the problems considered show that the fiducial solutions and the approximate $Z$ fiducial solutions are satisfactory when the sample sizes are not too small.

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## Appendix

The following R code can be used to generate a sample from the fiducial distribution of $M_x$ described in Section 2.

### *R code*

```
Mx = c()
sq seq(x,x Nx-nx,1) # support of the fiducial distribu-
tion of Mx
ps0 phyper(x-1, sq, Nx-sq, nx); ps1 phyper(x, sq, Nx-sq,
nx)
u runif(N) # N number of uniform variates
for(j in 1:N){
ind which(ps0 u[j] & u[j]  ps1)
if(length(ind)  1){
Mx[j]  sq[ind]}
else{
Mx[j]  sample(sq[ind],1)}}
```

The sample $M_x[1],\ldots,M_x[N]$ is a simulated sample from the fiducial distribution of $M_x$. Fiducial inference on $M_x$ can be obtained using the fiducial sample.