3.5 Problems and Solutions (last updated 24 September 2018)

1. Two family vehicles, a car and a truck, are classified by region of assembly. The three assembly regions are North America (N), Europe (E) and Asia (A). We will refer to vehicles assembled in North America as domestic and those assembled in Europe or Asia as foreign.

a. List the 9 elementary outcomes for this experiment (Explain your notation.).

b. List the outcomes in the event that one of the vehicles is domestic and the other is foreign.

c. List the outcomes in the event that at least one of the vehicles is foreign.

d. List the outcomes in the complement of the event in (c).

Solution

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a. Let $\Omega = \{(x_1, x_2) : x_1, x_2 \in \{N, E, A\}\}$ where x_1 denotes the region of assembly for the car and x_2 denotes the region of assembly for the truck. Then the 9 elementary outcomes in Ω are

$\Omega = \{(N, N), (N, E), (N, A), (E, N), (E, E), (E, A), (A, N), (A, E), (A, A)\}$

b. The event that one of the vehicles is domestic and the other is foreign is

$\{(N, E), (N, A), (E, N), (A, N)\}$

c. The event that at least one of the vehicles is foreign is

$\{(N, E), (N, A), (E, N), (E, E), (E, A), (A, N), (A, E), (A, A)\}$

d. The complement of the event in (c) is $\{(N, N)\}$

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2. Each of four restaurants is classified as local (L) or in a national/regional chain (N).

a. List are the 16 elementary outcomes for this experiment (Explain your notation.).

b. List the outcomes in the event that exactly three of the restaurants are local.

c. List the outcomes in the event that all four restaurants are of the same type.

d. List the outcomes in the event that at most one of the four restaurants are local.

e. List the outcomes in the union events in (c) and (d), and list the outcomes in the intersection of these two events.

f. List the outcomes in the union events in (b) and (c), and list the outcomes in the intersection of these two events.

g. List the outcomes in the complement of the event in (d)?

Solution

a. Let $\Omega = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \{L, N\}\}$ where x_i denotes the classification of the i^{th} restaurant. Thus the elementary outcomes in Ω are
$\Omega = \{(L, L, L, L), (L, L, L, N), (L, L, N, L), (L, L, N, N), (L, N, L, L), (L, N, L, N), (L, N, N, L), (L, N, N, N), (N, L, L, L), (N, L, L, N), (N, L, N, L), (N, L, N, N), (N, N, L, L), (N, N, L, N), (N, N, N, L), (N, N, N, N)\}$

b. The event that exactly three of the restaurants are local is
$\{(L, L, L, N), (L, L, N, L), (L, N, L, L), (N, L, L, L)\}$

c. The event that all four restaurants are of the same type is $\{(L, L, L, L), (N, N, N, N)\}$

d. The event that at most one of the four restaurants are local is
$\{(L, N, N, N), (N, L, N, N), (N, N, L, N), (N, N, N, L), (N, N, N, N)\}$

e. The union of the events in (c) and (d) is $\{(L, L, L, L), (N, N, N, N), (L, N, N, N), (N, L, N, N), (N, N, L, N), (N, N, N, L)\}$ The intersection of the events in (c) and (d) is $\{(N, N, N, N)\}$

f. The union of the events in (b) and (c) is $\{(L, L, L, N), (L, L, N, L), (L, N, L, L), (N, L, L, L), (L, L, L, L), (N, N, N, N)\}$ The intersection of the events in (b) and (c) is empty.

g. The complement of the event in (d) is $\{(L, L, L, L), (L, L, L, N), (L, L, N, L), (L, L, N, N), (L, N, L, L), (L, N, L, N), (L, N, N, L), (N, L, L, L), (N, L, L, N), (N, L, N, L), (N, N, L, L)\}$

3. A large construction company is currently working on three large projects (1,2,3). For $i = 1, 2, 3$, let A_i denote the event that project i is completed by its contract date. For each of the events described below, use the operations of unions, intersections, and complements to describe the event in terms of A_1, A_2, and A_3, draw a Venn diagram with three circles (similar to that above), and shade the region corresponding to the event.

a. At least one project is completed by its contract date.

b. All three projects are completed by their contract dates.

c. Only project 1 is completed by its contract date.

d. Exactly one project is completed by its contract date.

e. Project 1 is completed by its contract date or both of the other projects (but not all three) are completed by its contract date.

Solution

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a. At least one project is completed by its contract date means that project 1 is completed by its contract date, or project 2 is completed by its contract date, or project 3 is completed by its contract date. This is the union $A_1 \cup A_2 \cup A_3$.

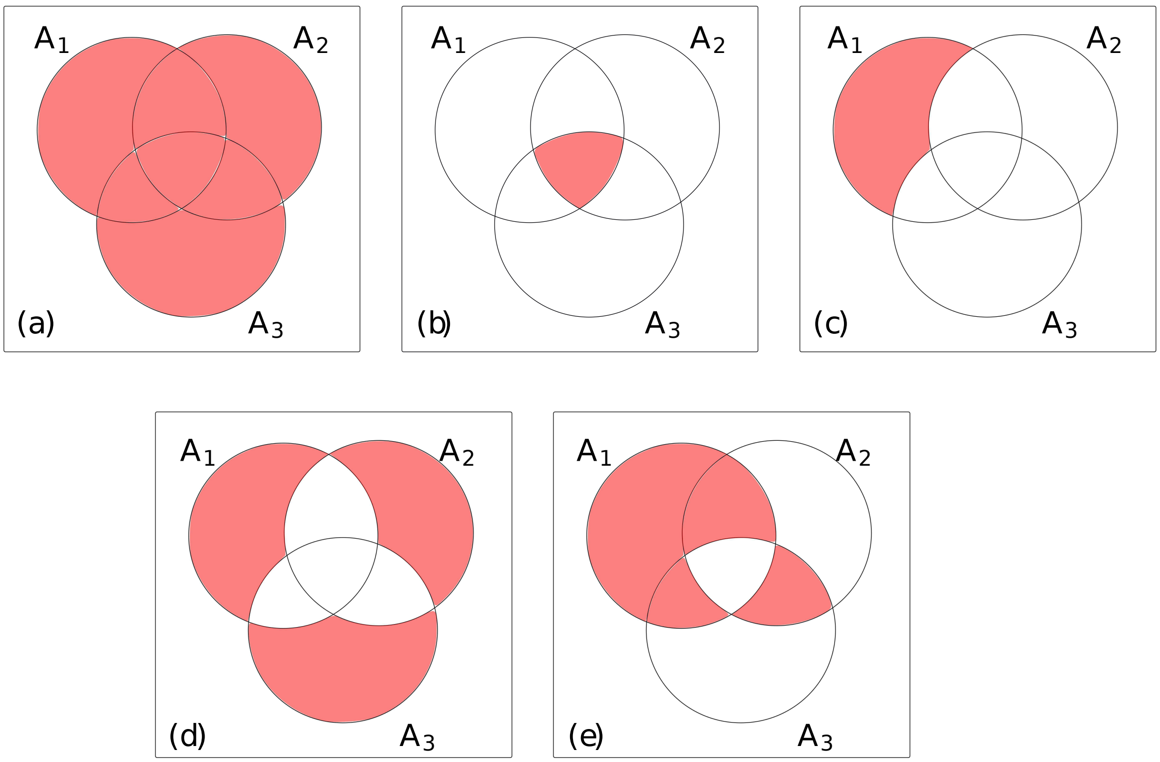
b. All three projects are completed by their contract dates means that project 1 is completed by its contract date, and project 2 is completed by its contract date, and project 3 is completed by its contract date. This is the intersection $A_1 \cap A_2 \cap A_3$.

c. Only project 1 is completed by its contract date means that project 1 is completed by its contract date while projects 2 and 3 are not completed by their contract dates. This can be expressed as $A_1 \cap A_2^c \cap A_3^c$.

d. Exactly one project is completed by its contract date means that: project 1 is completed by its contract date while projects 2 and 3 are not completed by their contract dates; or, project 2 is completed by its contract date while projects 1 and 3 are not completed by their contract dates; or, project 3 is completed by its contract date while projects 1 and 2 are not completed by their contract dates. This can be expressed as
$(A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c) \cup (A_1^c \cap A_2^c \cap A_3)$.

e. Project 1 is completed by its contract date or both of the other projects (but not all three) are completed by their contract dates means that project 1 is completed by its contract date while neither of projects 2 and 3 is completed by its contract date, or projects 1 and 2 are completed by their contract dates while project 3 is not completed by its contract date, or projects 1 and 3 are completed by their contract dates while project 2 is not completed by its contract date, or project 1 is not completed by its contract date while both projects 2 and 3 are completed by their contract dates. This can be expressed as

$(A_1 \cap A_2 \cap A_3^c) \cup (A_1 \cap A_2^c \cap A_3) \cup (A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3)$



4. Suppose that a die is tossed twice. Express the following events as subsets of the set of 36 ordered pairs (see Table 3.1 reproduced below) in the sample space for this experiment. If possible describe the location of the ordered pairs instead of listing them, *e.g.*, the outcomes in column two correspond to observing a two on the second toss.

a. A – the number on the first toss is larger than the number on the second toss.

b. B – the number on the first toss is even.

c. C – the first toss yields a four.

d. D – the sum of the numbers on the two tosses is seven

e. $A \cup C$

f. $A \cap D$

g. C^c

Solution

a. A contains the pairs in the lower left triangle below the $(1,1),(2,2),\ldots,(6,6)$ diagonal.

$A = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3),(6,4),(6,5)\}$

b. B contains the pairs in rows 2, 4, and 6.

$B = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

c. C contains the pairs in row 4. $C = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$

d. D contains the pairs on the lower left to upper right diagonal $(6,1),(5,2),\ldots,(1,6)$ diagonal. $D = \{(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)\}$

e. $A \cup C$ contains the pairs in the lower left triangle below the $(1,1),(2,2),\ldots,(6,6)$ diagonal plus the other pairs in row 4.

$A \cup C = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3),(6,4),(6,5),(4,4),(4,5),(4,6)\}$

f. $A \cap D$ contains the pairs on the lower left to upper right diagonal $(6,1),(5,2),\ldots,(1,6)$ diagonal for which the first element is larger than the second.

$A \cap D = \{(6,1),(5,2),(4,3)\}$

g. C^c contains the pairs in rows 1, 2, 3, 5, and 6.

$C^c = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

5. A vehicle arriving at an intersection can turn left, turn right, or go forward through the intersection. For the following questions let L denote "turns left", R denote "turns right", and F denote "goes forward through the intersection". Consider an experiment consisting of observing the movement of two vehicles through the intersection.

a. List the elementary outcomes in the sample space for this experiment.

b. List the outcomes in the event that both vehicles turn.

c. List the outcomes in the event that the vehicles turn in opposite directions.

d. List the outcomes in the event that the first vehicle turns left.

e. List the outcomes in the union events in (c) and (d), and list the outcomes in the intersection of these two events?

Solution

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a. Let $\Omega = \{(x_1, x_2) : x_1, x_2 \in \{L, R, F\}\}$ where x_1 denotes the movement of the first vehicle and x_2 denotes the movement of the second vehicle. Thus the elementary outcomes in Ω are $\Omega = \{(L, L), (L, R), (L, F), (R, L), (R, R), (R, F), (F, L), (F, R), (F, F)\}$

b. The event "both vehicles turn" (denote this event by A) contains the outcomes for which neither vehicle goes forward. Thus $A = \{(L, L), (L, R), (R, L), (R, R)\}$.

c. The event "the vehicles turn in opposite directions" (denote this event by B) contains the outcomes for which one vehicle turns left and the other turns right. Thus $B = \{(L, R), (R, L)\}$.

d. The event "the first vehicle turns left" (denote this event by C) contains the outcomes for which the first vehicle turns left. Thus $C = \{(L, L), (L, R), (L, F)\}$.

e. The event $B \cup C$ (the union of the events in (c) and (d)) is the event that "the vehicles turn in opposite directions" or "the first vehicle turns left". Thus $B \cup C = \{(L, R), (R, L), (L, F), (L, L)\}$.

The event $B \cap C$ (the intersection of the events in (c) and (d)) is the event that "the vehicles turn in opposite directions" and "the first vehicle turns left". Thus $B \cap C = \{(L, R)\}$.

extra The event $A \cup C$ (the union of the events in (b) and (d)) is the event that "both vehicles turn" or "the first vehicle turns left". Thus $A \cup C = \{(L, L), (L, R), (R, L), (R, R), (L, F)\}$.
The event $A \cap C$ (the intersection of the events in (b) and (d)) is the event that "both vehicles turn" and "the first vehicle turns left". Thus $A \cap C = \{(L, L), (L, R)\}$.

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6. Suppose that one card is selected from a deck of 20 cards that contains 10 red cards numbered from 1 to 10 and 10 blue cards numbered from 1 to 10.

Solution

a. List 20 elementary outcomes in the sample space Ω of this experiment.

$$\Omega = \{R_1, R_2, \ldots, R_{10}, B_1, B_2, \ldots, B_{10}\}$$

b. List the outcomes in the event A "a card with an even number is selected".

$$A = \{R_2, R_4, R_6, R_8, R_{10}, B_2, B_4, B_6, B_8, B_{10}\}$$

c. List the outcomes in the event B "a blue card is selected".

$$B = \{B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}\}$$

d. List the outcomes in the event C "a card with a number less than five is selected".

$$C = \{R_1, R_2, R_3, R_4, B_1, B_2, B_3, B_4\}$$

Describe each of the following events both in words and as a subset of Ω:

e. $A \cap B \cap C = \{B_2, B_4\}$ The card is even, and blue, and less than 5

f. $A \cup B \cup C = \{R_1, R_2, R_3, R_4, R_6, R_8, R_{10}, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}\}$ The card is even or blue or less than five

g. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = \{R_2, R_4, B_2, B_4, B_6, B_8, B_{10}\}$ The card is even and blue or the card is even and less than five

h. $B \cap C^c = \{B_5, B_6, B_7, B_8, B_9, B_{10}\}$ The card is blue and greater than or equal to 5

i. $A^c \cap B^c \cap C^c = \{R_5, R_7, R_9\}$ The card is odd and red and greater than or equal to 5