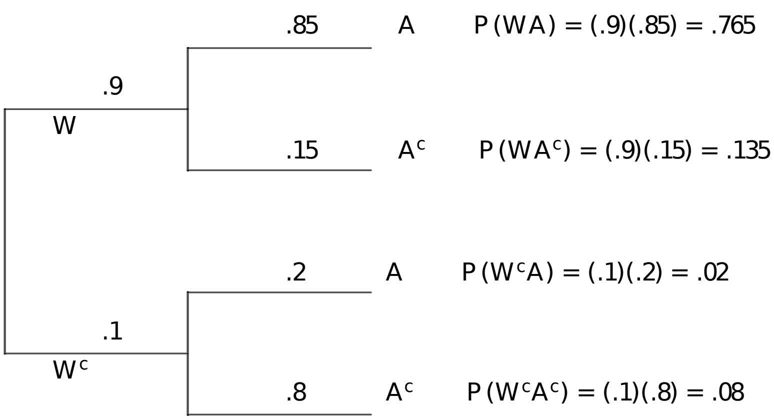
6.4 Problems and Solutions

(last update 30 May 2018)

1. You ask your neighbor to water a sickly plant while you are on vacation. Without water, it will die with probability .8; with water, it will die with probability .15. You are 90 percent certain that your neighbor will remember to water the plant.
(a) What is the probability that the plant will be alive when you return?
(b) If the plant is dead upon your return, what is the probability that your neighbor forgot to water it?

Solution



Let W denote the event that the neighbor waters the plant and let A denote the event that the plant is alive when you return. We know that, $P(A^c|W^c) = .8$, $P(A^c|W) = .15$, and $P(W) = .9$. Thus

$P(AW) = P(A|W)P(W) = (1 - P(A^c|W))P(W) = (1 - .15)(.9) = (.85)(.9) = .765$

$P(AW^c) = P(A|W^c)P(W^c) = (1-P(A^c|W^c))(1-P(W)) = (1-.8)(1-.9) = (.2)(.1) = .02$

$P(A^cW) = P(A^c|W)P(W) = (.15)(.9) = .135$

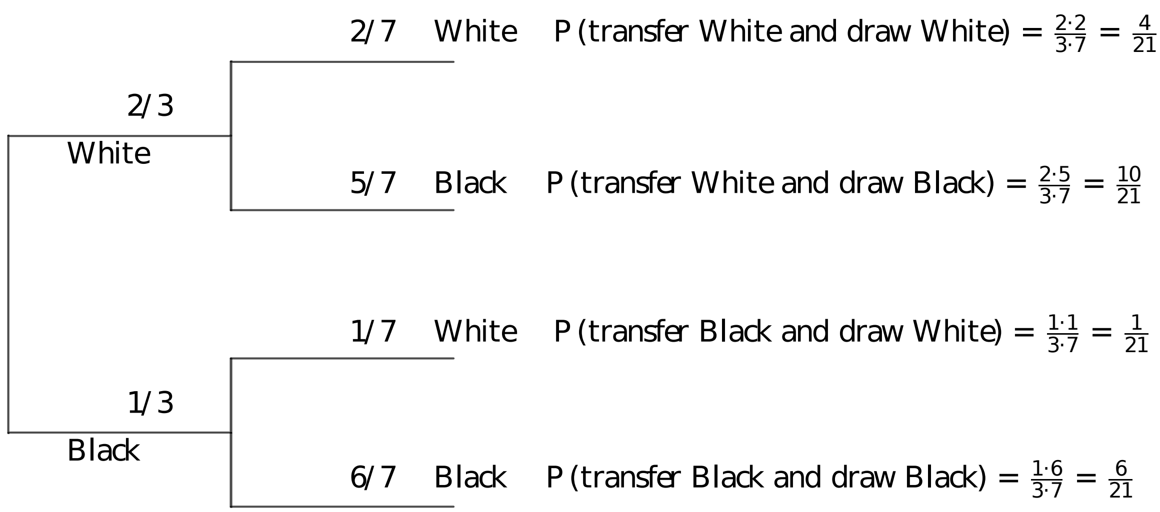
$P(A^cW^c) = P(A^c|W^c)P(W^c) = (.8)(1 - .9) = (.8)(.1) = .08$

(a) $$P(A) = P(AW) + P(AW^c) = (.85)(.9) + (.2)(.1) = .765 + .02 = .785$$

(b) $$P(W^c|A^c) = \frac{P(A^cW^c)}{P(A^c)} = \frac{P(A^cW^c)}{1 - P(A)} = \frac{(.8)(.1)}{1 - .785} = \frac{.08}{.215} = \frac{80}{215} \approx .3721$$

2. Box A contains 2 white balls and 1 black ball, whereas box B contains 1 white ball and 5 black balls. A ball is drawn at random from box A and transferred to box B. A ball is then drawn from the seven balls in box B. Given that the ball drawn from box B is white, what is the conditional probability that the ball transferred from box A to box B was white?

Solution



Since there are 2 white balls and 1 black ball in box A, we have $P(\text{transfer white}) = 2/3$ and $P(\text{transfer black}) = 1/3$.
If a white ball is transferred, then there are 2 white balls and 5 black balls in box B. Thus $P(\text{draw white from } B|\text{transfer white}) = \frac{2}{7}$ and $P(\text{transfer white and draw white}) = \frac{2}{3} \cdot \frac{2}{7}$.
If a black ball is transferred, then there is 1 white ball and 6 black balls in box B. Thus $P(\text{draw white from } B|\text{transfer black}) = \frac{1}{7}$ and $P(\text{transfer black and draw white}) = \frac{1}{3} \cdot \frac{1}{7}$.
Combining these facts yields

$$P(\text{transfer white}|\text{draw white from } B) = \frac{\frac{2}{3}\cdot\frac{2}{7}}{\frac{2}{3}\cdot\frac{2}{7}+\frac{1}{3}\cdot\frac{1}{7}} = \frac{4}{5}.$$

3. Maria will take two books with her on a trip. Suppose that the probability that she will like book 1 is .6, the probability that she will like book 2 is .5, and the probability that she will like both books is .4. Find the conditional probability that she will like book 2 given that she did not like book 1.

Solution

Let A denote the event that Maria will like book 1 and let B denote the event that she will like book 2. We know that $P(A) = .6$, $P(B) = .5$, and $P(AB) = .4$. Thus $P(A^cB) = P(B) - P(AB) = .5 - .4 = .1$ and the conditional probability that she will like book 2 given that she did not like book 1 is

$$P(B|A^c) = \frac{P(A^cB)}{P(A^c)} = \frac{.1}{1 - .6} = .25.$$

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4. There is a 60 percent chance that event A will occur. If A does not occur, then there is a 10 percent chance that B will occur. What is the probability that at least one of the events A or B will occur?

Solution

We know that $P(A) = .6$ and $P(B|A^c) = .1$. Since $A \cup B = A \cup A^cB$ and A and A^cB are disjoint, we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(A^cB) = P(A) + P(A^c)P(B|A^c) \\ &= P(A) + (1 - P(A))P(B|A^c) = .6 + (.4)(.1) = .64. \end{aligned}$$

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5. Six balls are to be randomly chosen from a box containing 8 red, 10 green, and 12 blue balls.

First assume that the balls are chosen at random with replacement.

(a) What is the probability at least one red ball is chosen?

(b) Given that no red balls are chosen, what is the conditional probability that there are exactly 2 green balls among the 6 chosen?

Next assume that the balls are chosen at random without replacement.

(c) What is the probability at least one red ball is chosen?

(d) Given that no red balls are chosen, what is the conditional probability that there are exactly 2 green balls among the 6 chosen?

Solution

Let R denote the number of red balls in the sample, let G denote the number of green balls in the sample, and let B denote the number of blue balls in the sample

(a) $$P(R \geq 1) = 1 - P(R = 0) = 1 - \binom{6}{0}\left(\frac{8}{30}\right)^0\left(\frac{22}{30}\right)^6 \approx .8445$$

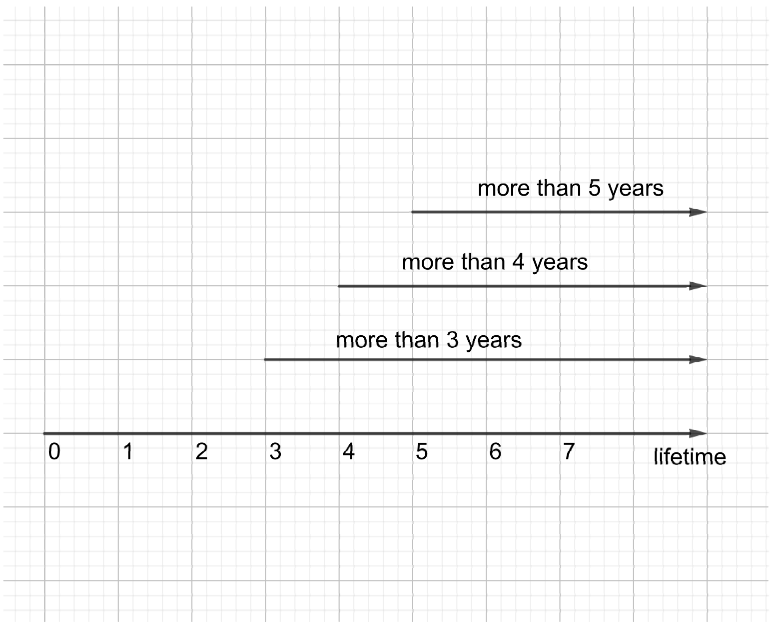
(b)
$$\begin{aligned}
P(G = 2|R = 0) &= \frac{P(R = 0 \text{ and } G = 2)}{P(R = 0)} \\
&= \frac{P(R = 0 \text{ and } G = 2 \text{ and } B = 4)}{P(R = 0)} \\
&= \frac{\binom{6}{2}\left(\frac{8}{30}\right)^0\left(\frac{10}{30}\right)^2\left(\frac{12}{30}\right)^4}{\binom{6}{0}\left(\frac{8}{30}\right)^0\left(\frac{22}{30}\right)^6} \\
&= \binom{6}{2}\left(\frac{10}{22}\right)^2\left(\frac{12}{22}\right)^4 \approx .2743
\end{aligned}$$

(c) $$P(R \geq 1) = 1 - P(R = 0) = 1 - \frac{\binom{8}{0}\binom{22}{6}}{\binom{30}{6}} \approx .8743$$

(d)
$$\begin{aligned}
P(G = 2|R = 0) &= \frac{P(R = 0 \text{ and } G = 2)}{P(R = 0)} \\
&= \frac{P(R = 0 \text{ and } G = 2 \text{ and } B = 4)}{P(R = 0)} \\
&= \frac{\frac{\binom{8}{0}\binom{10}{2}\binom{12}{4}}{\binom{30}{6}}}{\frac{\binom{8}{0}\binom{22}{6}}{\binom{30}{6}}} \\
&= \frac{\binom{10}{2}\binom{12}{4}}{\binom{22}{6}} \approx .2985
\end{aligned}$$

6. The probability that a new car battery functions for more than 3 years is .8, the probability that it functions for more than 4 years is .4, and the probability that it functions for more than 5 years miles is .1.
If a new car battery is still working after 3 years, what is the probability that
(a) its total life will exceed 4 years?
(b) its additional life will exceed 2 years?

Solution



Let A_3 denote the event that the battery functions for more than 3 years, let A_4 denote the event that the battery functions for more than 4 years, and let A_5 denote the event that the battery functions for more than 5 years. Notice that these events are nested with $A_5 \subset A_4 \subset A_3$. Therefore, $A_4A_3 = A_4$ and $A_5A_4 = A_5$. We know that $P(A_3) = .8$, $P(A_4) = .4$, and $P(A_5) = .1$.

(a) $P(\text{total life will exceed 4 years} \mid \text{still working after 3 years}) = P(A_4|A_3)$ and

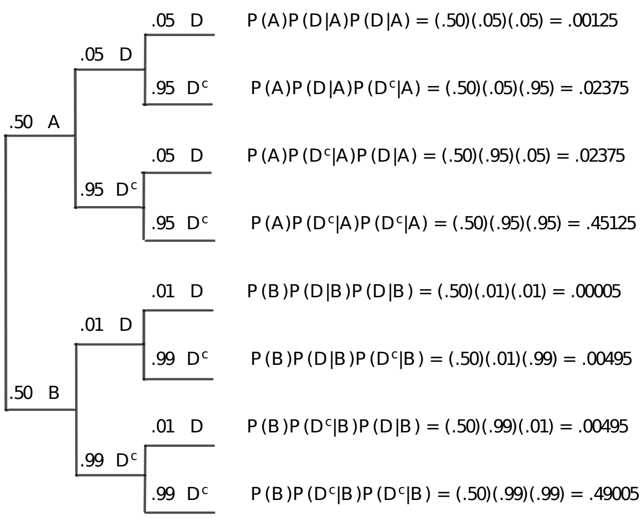
$$P(A_4|A_3) = \frac{P(A_4A_3)}{P(A_3)} = \frac{P(A_4)}{P(A_3)} = \frac{.4}{.8} = .5.$$

(b) $P(\text{additional life will exceed 2 years} \mid \text{still working after 3 years}) = P(A_5|A_3)$ and

$$P(A_5|A_3) = \frac{P(A_5A_3)}{P(A_3)} = \frac{P(A_5)}{P(A_3)} = \frac{.1}{.8} = .125.$$

7. Two factories, A and B, produce radios. Each radio produced at factory A is defective with probability .05, whereas each one produced at factory B is defective with probability .01. Suppose you purchase two radios that were produced at the same factory, which is equally likely to have been either factory A or factory B. If the first radio is defective, what is the conditional probability that the other one is also defective?

Solution



P(first defective) = .00125 + .02375 + .00005 + .00495 = .03
P(both defective) = .00125 + .00005 = .0013

Let A denote the event that a radio is produced at factory A, let B denote the event that a radio is produced at factory B, and let D denote the event that a radio is defective. We know that $P(A) = P(B) = .5$, $P(D|A) = .05$, and $P(D|B) = .01$. Assuming that the two radios are selected at random and independently from radios produced at the same factory, we have

$$\begin{aligned}
P(\text{second defective} \mid \text{first defective}) &= \frac{P(\text{both defective})}{P(\text{first defective})} \\
&= \frac{P(A)P(D|A)P(D|A) + P(B)P(D|B)P(D|B)}{P(A)P(D|A) + P(B)P(D|B)} \\
&= \frac{(.5)(.05)^2 + (.5)(.01)^2}{(.5)(.05) + (.5)(.01)} \\
&= \frac{(.05)^2 + (.01)^2}{(.05) + (.01)} = \frac{13}{300} \approx .0433
\end{aligned}$$

8. Consider a game of bridge.

First assume that West has no aces.

(a) What is the conditional probability that his partner (East) has no aces?

(b) What is the conditional probability that his partner (East) has 2 or more aces?

Next assume that West has exactly 1 ace.

(c) What is the conditional probability that his partner (East) has no aces?

(d) What is the conditional probability that his partner (East) has 2 or more aces?

Solution

Let E denote the number of aces in East's hand and let W denote the number of aces in West's hand. For the following computations, think of choosing four positions for the aces out of the 52 positions in the deal; and think of the 52 cards in the deal as being partitioned into a set of 13 for East, a set of 13 for West, and a set of 26 for North and South.

$$(a) \qquad P(E=0|W=0) = \frac{P(E=0 \text{ and } W=0)}{P(W=0)} = \frac{\frac{\binom{13}{0}\binom{13}{0}\binom{26}{4}}{\binom{52}{4}}}{\frac{\binom{13}{0}\binom{39}{4}}{\binom{52}{4}}} = \frac{\binom{13}{0}\binom{26}{4}}{\binom{39}{4}} \approx .1818$$

$$(b) \qquad \begin{aligned} P(E \geq 2 \text{ and } W=0) &= \frac{P(E \geq 2 \text{ and } W=0)}{P(W=0)} \\ &= \frac{\frac{\binom{13}{0}\binom{13}{2}\binom{26}{2}+\binom{13}{0}\binom{13}{3}\binom{26}{1}+\binom{13}{0}\binom{13}{4}\binom{26}{0}}{\binom{52}{4}}}{\frac{\binom{13}{0}\binom{39}{4}}{\binom{52}{4}}} \\ &= \frac{\binom{13}{2}\binom{26}{2}+\binom{13}{3}\binom{26}{1}+\binom{13}{4}\binom{26}{0}}{\binom{39}{4}} \approx .4073 \end{aligned}$$

$$(c) \qquad P(E=0|W=1) = \frac{P(E=0 \text{ and } W=1)}{P(W=1)} = \frac{\frac{\binom{13}{1}\binom{13}{0}\binom{26}{3}}{\binom{52}{4}}}{\frac{\binom{13}{1}\binom{39}{3}}{\binom{52}{4}}} = \frac{\binom{13}{0}\binom{26}{3}}{\binom{39}{3}} \approx .2845$$

(d)

$$P(E \geq 2 \text{ and } W = 1) = \frac{P(E \geq 2 \text{ and } W = 1)}{P(W = 1)}$$

$$= \frac{\frac{\binom{13}{1}\binom{13}{2}\binom{26}{1}+\binom{13}{1}\binom{13}{3}\binom{26}{0}}{\binom{52}{4}}}{\frac{\binom{13}{1}\binom{39}{3}}{\binom{52}{4}}}$$

$$= \frac{\binom{13}{2}\binom{26}{1}+\binom{13}{3}\binom{26}{0}}{\binom{39}{3}} \approx .2532$$

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9. Suppose that one card is selected at random from a standard deck of 52 cards. Let A denote the event that the card is a spade, and let B denote the event that the card is a king. Are A and B independent?
Let C denote the event that the card is red (a heart or diamond), and, as above, let B denote the event that the card is a king. Are C and B independent?
Let D denote the event that the card is a face card (a jack, queen, or king), and, as above, let B denote the event that the card is a king. Are D and B independent?

Solution

First note that:
$P(A) = \frac{13}{52} = \frac{1}{4}$, $P(B) = \frac{4}{52} = \frac{1}{13}$, $P(C) = \frac{26}{52} = \frac{1}{2}$, and $P(D) = \frac{12}{52} = \frac{3}{13}$.

Since AB is the event that the card is the king of spades (a king and a spade) and
$P(AB) = \frac{1}{52} = \frac{1}{4} \cdot \frac{1}{13} = P(A)P(B)$, we see that A and B are independent.

Since BC is the event that the card is the king of hearts or the king of diamonds (a king and red) and
$P(BC) = \frac{2}{52} = \frac{1}{26} = \frac{1}{13} \cdot \frac{1}{2} = P(B)P(C)$, we see that B and C are independent.

Since BD is the event that the card is a king (a face card and a king) and
$P(BD) = \frac{4}{52} = \frac{1}{13} \neq \frac{1}{13} \cdot \frac{3}{13} = P(B)P(D)$, we see that B and D are not independent.

10. Suppose that a fair coin is tossed twice. Let A denote the event that the first toss results in a head, let B denote the event that the second toss results in a tail, let C denote the event that both tosses result in a head, and let D denote the event that the results of the two tosses are not the same. Are A and B independent? Are A and C independent? Are A and D independent? Are B and C independent? Are B and D independent? Are C and D independent?

Solution

First note that:

$P(A) = P(\{HH, HT\}) = \frac{1}{2}$, $P(B) = P(\{HT, TT\}) = \frac{1}{2}$,
$P(C) = P(\{HH\}) = \frac{1}{4}$, and $P(D) = P(\{HT, TH\}) = \frac{1}{2}$.

$P(AB) = P(\{HT\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$ indicating that A and B are independent.
$P(AC) = P(\{HH\}) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{4} = P(A)P(C)$ indicating that A and C are not independent.
$P(AD) = P(\{HT\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(D)$ indicating that A and D are independent.
$P(BC) = P(\emptyset) = 0 \neq \frac{1}{2} \cdot \frac{1}{4} = P(B)P(C)$ indicating that B and C are not independent.
$P(BD) = P(\{HT\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(B)P(D)$ indicating that B and D are independent.
$P(CD) = P(\emptyset) = 0 \neq \frac{1}{4} \cdot \frac{1}{2} = P(C)P(D)$ indicating that C and D are not independent.

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11. Suppose that a balanced die is tossed twice. Let A denote the number observed on the first toss, let B denote the number observed on the second toss, and let C denote the sum of the numbers obtained on the two tosses.
Are the events $[A = 3]$ and $[C = 6]$ independent?
Are the events $[A = 3]$ and $[C = 7]$ independent?
Are the events $[B = 4]$ and $[C = 6]$ independent?
Are the events $[B = 4]$ and $[C = 7]$ independent?
Are the events $[A$ is even$]$ and $[B = 3]$ independent?
Are the events $[A$ is even$]$ and $[C = 4]$ independent?
Are the events $[A$ is even$]$ and $[B$ is even$]$ independent?
Are the events $[A = 4]$, $[B = 3]$, and $[C = 7]$ independent? Are they pairwise independent?

Solution

$P([A = 3]) = P([A = 4]) = P([B = 3]) = P([B = 4]) = \frac{6}{36} = \frac{1}{6}$
$P([C = 7]) = \frac{6}{36} = \frac{1}{6}$
$P([C = 6]) = \frac{5}{36}$
$P([C = 4]) = \frac{3}{36} = \frac{1}{12}$

Since $[A=3] \cap [C=6] = \{(3,3)\}$,
$P([A=3] \cap [C=6]) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{5}{36} = P([A=3])P([C=6])$ and
$[A=3]$ and $[C=6]$ are not independent.

Since $[A=3] \cap [C=7] = \{(3,4)\}$,
$P([A=3] \cap [C=7]) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P([A=3])P([C=7])$ and
$[A=3]$ and $[C=7]$ are independent.

Since $[B=4] \cap [C=6] = \{(2,4)\}$,
$P([B=4] \cap [C=6]) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{5}{36} = P([B=4])P([C=6])$ and
$[B=4]$ and $[C=6]$ are not independent.

Since $[B=4] \cap [C=7] = \{(3,4)\}$,
$P([B=4] \cap [C=7]) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P([B=4])P([C=7])$ and
$[B=4]$ and $[C=7]$ are independent.

Since $[A \text{ is even}] \cap [B=3] = \{(2,3),(4,3),(6,3)\}$,
$P([A \text{ is even}] \cap [B=3]) = \frac{1}{12} = \frac{1}{2} \cdot \frac{1}{6} = P([A \text{ is even})P([B=3])$ and
$[A \text{ is even}]$ and $[B=3]$ are independent.

Since $[A \text{ is even}] \cap [C=4] = \{(2,2)\}$,
$P([A \text{ is even}] \cap [C=4]) = \frac{1}{36} \neq \frac{1}{2} \cdot \frac{1}{12} = P([A \text{ is even}])P([C=4])$ and
$[A \text{ is even}]$ and $[C=4]$ are not independent.

Since $[A \text{ is even}] \cap [B \text{ is even}] = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$,
$P([A \text{ is even}] \cap [B \text{ is even}]) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P([A \text{ is even}])P([B \text{ is even}])$ and
$[A \text{ is even}]$ and $[B \text{ is even}]$ are independent.

Note that $P([A=4]) = P([B=3]) = P([C=7]) = \frac{1}{6}$.
$[A=4] \cap [B=3] = \{(4,3)\}$, hence $P([A=4] \cap [B=3]) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$
$[A=4] \cap [C=7] = \{(4,3)\}$, hence $P([A=4] \cap [C=7]) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$
$[B=3] \cap [C=7] = \{(4,3)\}$, hence $P([B=3] \cap [C=7]) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$
$[A=4] \cap [B=3] \cap [C=7] = \{(4,3)\}$ hence $P([A=4] \cap [B=3] \cap [C=7]) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$

Therefore, $[A=4]$, $[B=3]$, and $[C=7]$ are not independent. But, they are pairwise independent.

12. Consider two boxes, say box A and box B. Suppose that box A contains 60 red balls and 40 white balls while box B contains 30 red balls and 70 white balls.

Suppose that a balanced die is tossed once. If the number on the die is one or two, then one ball is selected from box A and if it is three, four, five, or six, then one ball is selected from box B.
(a) Find the probability that the ball selected is red.

Now let's draw two balls from the box. As before, if the number on the die is one or two, we select from box A and if it is three, four, five, or six, then we select from box B. But this time, we select 2 balls at random with replacement.
(b) Find the probability that both of the two balls selected are red.
(c) Find the probability that exactly one of the two balls selected is red.

Now let's turn the question around. Suppose that we have completed this experiment and selected the two balls.
(d) Given that both of the two balls are red, what is the probability that these balls came from box A?
(e) Now suppose that exactly one of the two balls is red, what is the probability that these balls came from box A?

Solution

Box A contains 60 red balls and 40 white balls while box B contains 30 red balls and 70 white balls.

Let A denote the event that the ball or balls are selected from box A, let B denote that the ball or balls are selected from box B. We know that $P(A) = 1/3$ and $P(B) = 2/3$.

Let R_1 denote the event that a single ball selected is red, then $P(R_1|A) = 6/10$ and $P(R_1|B) = 3/10$. Thus

$$(a) \qquad \begin{aligned} P(R_1) &= P(A)P(R_1|A) + P(B)P(R_1|B) \\ &= \frac{1}{3} \cdot \frac{6}{10} + \frac{2}{3} \cdot \frac{3}{10} = \frac{2}{5} \end{aligned}$$

Now let R denote the number of red balls among the two balls selected with replacement.

(b)

$$\begin{aligned} P(R=2) &= P(A)P(R=2|A) + P(B)P(R=2|B) \\ &= \frac{1}{3} \cdot \left(\frac{6}{10}\right)^2 + \frac{2}{3} \cdot \left(\frac{3}{10}\right)^2 = \frac{36+18}{300} = \frac{9}{50} \end{aligned}$$

(c)

$$\begin{aligned} P(R=1) &= P(A)P(R=1|A) + P(B)P(R=1|B) \\ &= \frac{1}{3} \cdot \binom{2}{1} \cdot \frac{6}{10} \cdot \frac{4}{10} + \frac{2}{3} \cdot \binom{2}{1} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{48+84}{300} = \frac{11}{25} \end{aligned}$$

(d)

$$\begin{aligned} P(A|R=2) &= \frac{P(A \cap [R=2])}{P(R=2)} \\ &= \frac{P(A)P(R=2|A)}{P(R=2)} = \frac{\frac{36}{300}}{\frac{36+18}{300}} = \frac{36}{36+18} = \frac{2}{3} \end{aligned}$$

(e)

$$\begin{aligned} P(A|R=1) &= \frac{P(A \cap [R=1])}{P(R=1)} \\ &= \frac{P(A)P(R=1|A)}{P(R=1)} = \frac{\frac{48}{300}}{\frac{48+84}{300}} = \frac{48}{48+84} = \frac{4}{11} \end{aligned}$$

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13. Suppose that a free medical test for a certain disease is available. This test is 90% reliable in the following sense: If a person has the disease, there is a probability of .9 that the test will correctly indicate that the person has the disease (a true positive); whereas, if a person does not have the disease, there is a probability of .1 that the test will incorrectly indicate that the person has the disease (a false positive). In addition, suppose that this disease is rare in the sense that there is only a 1 in 10,000 chance that you have the disease. Suppose that, on a whim, you decide to take the test and when the results come back the test indicates that you have the disease! With this additional information how much does the probability that you actually have the disease increase? In other words, how does the conditional probability that you have the disease given that the test indicates that you have the disease compare to the unconditional probability that you have the disease?

Solution

Let D denote the event that you have the disease and let T denote the event that you test positive for the disease. We know that $P(D) = \frac{1}{10{,}000}$, $P(T|D) = .9$, and $P(T|D^c) = .1$. Combining this information we find that
$P(TD) = P(D)P(T|D) = \frac{9}{100{,}000}$ and
$P(TD^c) = P(D^c)P(T|D^c) = \frac{9{,}999}{100{,}000}$. Hence,

$$P(D|T) = \frac{P(TD)}{P(T)} = \frac{\frac{9}{100{,}000}}{\frac{9+9{,}999}{100{,}000}} = \frac{9}{10,008}$$

Notice that, since the disease is rare, the conditional probability that you have the disease given that the test indicates that you have the disease is still small. But, the conditional probability that you have the disease given that the test indicates that you have the disease is about 9 times the unconditional probability that you have the disease
($\frac{9}{10{,}008}$ versus $\frac{1}{10{,}000}$).