8.5 Problems and Solutions (last update 1 November 2024)

For each of the questions below:

i) Define the random variable. That is, identify the quantity being modeled and define the corresponding random variable. For example, in the NHANES example in the book we measured the heights of a random sample of adult women in the United States so we would say let X denote the height of an adult woman in the United States. Then express the answer in symbolic form as a function of X. For example, the probability that the height of a randomly selected adult woman in the United States exceeds 60 inches can be expressed as $P(X > 60)$.

ii) Sketch a normal density curve, label the X–axis with the value of the mean and the values relevant for the question, and shade the relevant area or areas under the density curve.

iii) Compute the requested numerical value or values to 4 decimal place accuracy.

You can use GeoGebra to do normal probability calculations and to create normal density curves.

You can also use a TI–84 (and similar) calculator to do normal probability calculations.
`normalcdf`$(a, b, \mu, \sigma) = P(a < X \leq b)$ when X follows a normal distribution with mean μ and standard deviation σ.
`invNorm`(p, μ, σ) gives the value of a such that $P(X \leq a) = p$ when X follows a normal distribution with mean μ and standard deviation σ.

When a problem asks how would you characterize the largest 5% of the observations, it is asking you to determine the value of a for which $P(X \geq a) = .05$

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1. The birth weight of babies born in the normal range of 37–43 weeks gestational age in the US can be reasonably modeled using a normal distribution with mean 7.7 pounds and standard deviation 1.3 pounds.

a) Find the probability that the birth weight of a randomly selected baby of this type exceeds 9.1 pounds.
b) Find the probability that the birth weight of a randomly selected baby of this type is between 6.5 and 9.1 pounds.
c) Find the probability that the birth weight of a randomly selected baby of this type is either less than 6 pounds or greater than 9 pounds.
d) How would you characterize the largest 5% of all such birth weights?
e) How would you characterize the smallest 10% of all such birth weights?

Solution

Let X denote the birth weight of a baby of the type described above. We can model the distribution of X using a normal distribution with mean $\mu = 7.7$ pounds and standard deviation $\sigma = 1.3$ pounds.

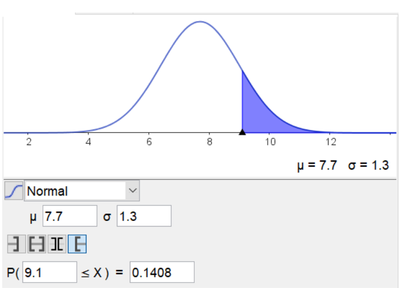
a) Find the probability that the birth weight of a randomly selected baby of this type exceeds 9.1 pounds.
In terms of X we want to find $P(X > 9.1)$. Using the calculator this is

$$P(X > 9.1) = \text{normalcdf}(9.1, 1E99, 7.7, 1.3) = .14076$$

If we standardize we have

$$\begin{aligned} P(X > 9.1) &= P(Z > \frac{9.1 - 7.7}{1.3}) \\ &= \text{normalcdf}(\frac{1.4}{1.3}, 1E99, 0, 1) = \text{normalcdf}(1.0769, 1E99, 0, 1) = .14076 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.



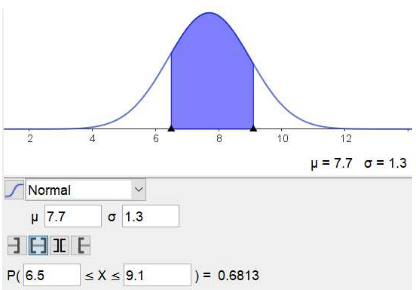
b) Find the probability that the birth weight of a randomly selected baby of this type is between 6.5 and 9.1 pounds.
In terms of X we want to find $P(6.5 < X < 9.1)$. Using the calculator this is

$$P(6.5 < X < 9.1) = \text{ normalcdf}(6.5, 9.1, 7.7, 1.3) = .68126$$

If we standardize we have

$$\begin{aligned} P(6.5 < X < 9.1) &= P(\frac{6.5 - 7.7}{1.3} < Z < \frac{9.1 - 7.7}{1.3}) \\ &= \text{ normalcdf}(\frac{-1.2}{1.3}, \frac{1.4}{1.3}, 0, 1) = .68126 \\ &= \text{ normalcdf}(-.923077, 1.0769, 0, 1) = .68126 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.



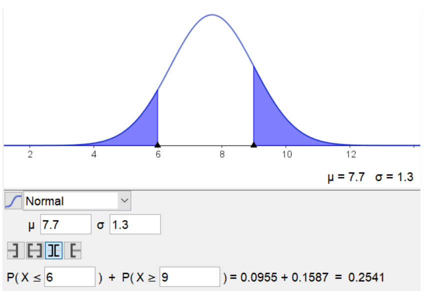
c) Find the probability that the birth weight of a randomly selected baby of this type is either less than 6 pounds or greater than 9 pounds.
In terms of X we want to find $P(X < 6 \text{ or } X > 9)$. Using the calculator this is

$$\begin{aligned} P(X < 6 \text{ or } X > 9) &= P(X < 6) + P(X > 9) \\ &= \text{normalcdf}(-1E99, 6, 7.7, 1.3) + \text{normalcdf}(9, 1E99, 7.7, 1.3) \\ &= .09549 + .15866 = .25415 \end{aligned}$$

If we standardize we have

$$\begin{aligned} P(X < 6 \text{ or } X > 9) &= P(Z < \frac{6-7.7}{1.3}) + P(Z > \frac{9-7.7}{1.3}) \\ &= \text{normalcdf}(-1E99, \frac{-1.7}{1.3}, 0, 1) + \text{normalcdf}(\frac{1.3}{1.3}, 1E99, 0, 1) \\ &= \text{normalcdf}(-1E99, -1.30769, 0, 1) + \text{normalcdf}(1, 1E99, 0, 1) \\ &= .09549 + .15866 = .25415 \end{aligned}$$

The figure and probability from GeoGebra are shown below.

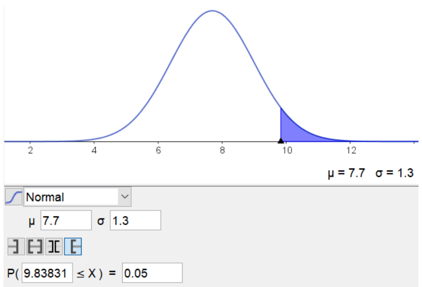


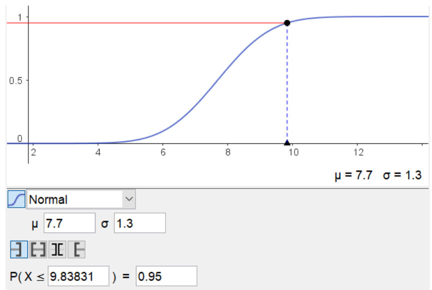
Notice that we could also do this computation by finding

$$P(X < 6) + P(X > 9) = 1 - P(6 < X < 9)$$

d) How would you characterize the largest 5% of all such birth weights?

Here we want to find the value of a for which $P(X > a) = .05$. The figures and probability expressions from GeoGebra below show the two ways you can do this. If the figure on the left I used the area under the pdf and entered .05. In the figure on the right I used the cdf, and set it equal to $1 - .05 = .95$. Both methods give $a = 9.83831$.





The approach of the figure on the right is easiest to implement with the calculator. There

$$a = \text{ invNorm}(.95, 7.7, 1.3) = 9.83831$$

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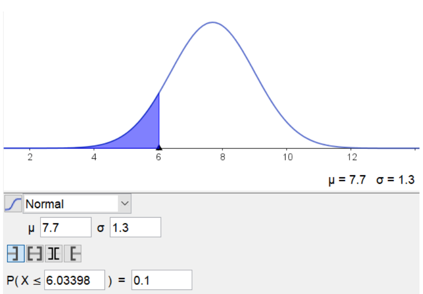
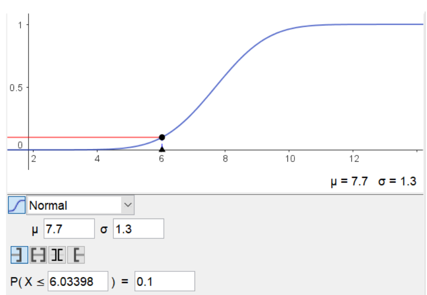
e) How would you characterize the smallest 10% of all such birth weights?

Here we want to find the value of a for which $P(X < a) = .10$. The figures and probability expressions from GeoGebra below show the two ways you can do this. If the figure on the left I used the area under the pdf and entered .10. In the figure on the right I used the cdf and set it equal to .10. Both methods give $a = 6.03398$.





The approach of the figure on the right is easiest to implement with the calculator. There

$$a = \text{ invNorm}(.10, 7.7, 1.3) = 6.03398$$

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2. A machine operation produces bearings whose diameters can be reasonably modeled as following a normal distribution with mean $\mu = .500$ inches and standard deviation $\sigma = .002$ inches.

a) Find the probability that the diameter of a randomly selected bearing of this type has an actual diameter of most .504 inches.
b) If specifications require that the bearing diameter be within .004 inches of .500, what fraction of the production will be unacceptable?
c) How would you characterize the largest 5% of the actual diameters of all the bearings produced using this machine?
d) Using .95 (95%) as a "large proportion of the time", find the range of actual diameters of all the bearings produced using this machine with the property that a value in this range will be selected with probability .95. That is, the range of values with the property that 95% of the bearings have diameters within this range.

Solution

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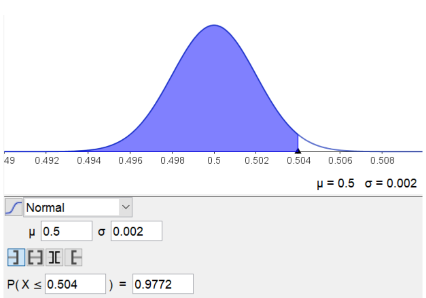
a) Find the probability that the diameter of a randomly selected bearing of this type has an actual diameter of at most .504 inches.
In terms of X we want to find $P(X \leq .504)$. Using the calculator this is

$$P(X \leq .504) = \text{normalcdf}(-1E99, .504, .500, .002) = .97725$$

If we standardize we have

$$P(X \leq .504) = P(Z \leq \frac{.504 - .500}{.002}) = \text{ normalcdf}(-1E99, 2, 0, 1) = .97725$$

The figure and probability from GeoGebra are shown in the screen capture below.



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b) If specifications require that the bearing diameter be within .004 inches of .500, what fraction of the production will be unacceptable?
In terms of X we want to find

$$\begin{aligned} P(|X - .500| > .004) &= P(X \le .500 - .004 \text{ or } X \ge .500 + .004) \\ &= 1 - P(.496 \le X \le .504) \end{aligned}.$$

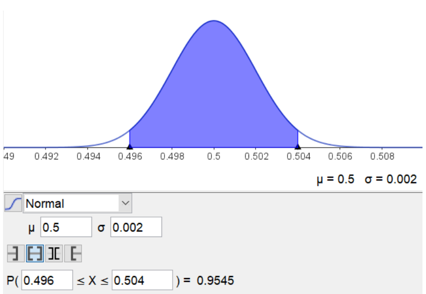
Using the calculator this is

$$\begin{aligned} 1 - P(.496 \le X \le .504) &= 1 - \text{normalcdf}(.496, .504, .5, .002) \\ &= 1 - .95450 = .04550 \end{aligned}$$

If we standardize we have

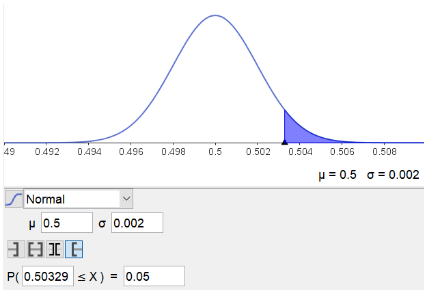
$$\begin{aligned} 1 - P(.496 \le X \le .504) &= 1 - P(\frac{.496 - .500}{.002} \le Z \le \frac{.504 - .500}{.002}) \\ &= 1 - \ \text{normalcdf}(\frac{-.004}{.002}, \frac{.004}{.002}, 0, 1) \\ &= 1 - \ \text{normalcdf}(-2, 2, 0, 1) = 1 - .95450 = .0455 \end{aligned}$$

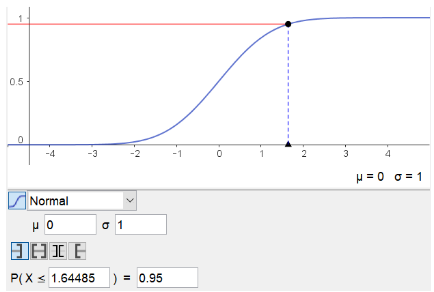
The figure and probability $P(.496 \le X \le .504)$ (the proportion that are acceptable) from GeoGebra are shown in the screen capture below.



c) How would you characterize the largest 5% of the actual diameters of all the bearings produced using this machine?

In terms of X we want to find a for which $P(X > a) = .05$. The figures and probability expressions from GeoGebra shown below show the two ways you can do this. If the figure on the left I used the area under the pdf and entered .05. In the figure on the right I used the cdf, and set it equal to $1 - .05 = .95$. Both methods give $a = .50329$.





The approach of the figure on the right is easiest to implement with the calculator. There

$$a = \text{ invNorm}(.95, .500, .002) = .50329$$

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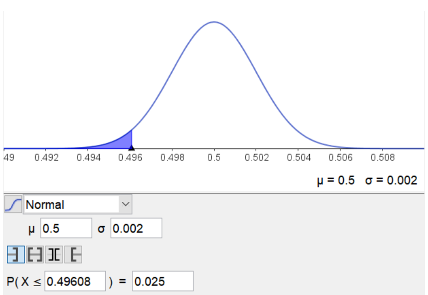
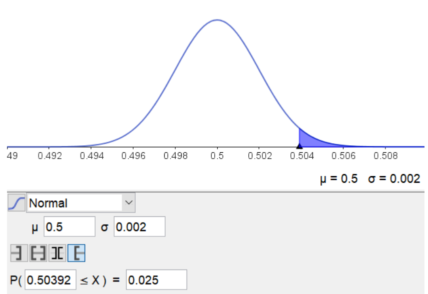
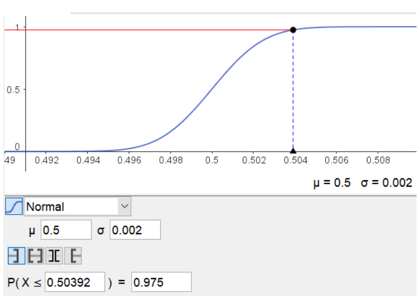
d) Using .95 (95%) as a "large proportion of the time", find the range of actual diameters of all the bearings produced using this machine with the property that a value in this range will be selected with probability .95. That is, the range of values with the property that 95% of the bearings have diameters within this range.
In terms of X we want to find values a and b (these are the endpoints of the interval we seek) for which $P(a \leq X \leq b) = .95$. It seems most reasonable to center the interval at $\mu = .500$ so that we have an interval of the form $.500 - c \leq X \leq .500 + c) = .95$, where c is a suitable constant. Since we know that $P(-1.96 \leq Z \leq 1.96) = .95$ and since this means that the probability that X is within 1.96 standard deviation units of its mean is .95, we see that $c = 1.96 \times .002 = .00392$ and the values we are looking for are $a = .500 - .00392 = .49608$ and $b = .500 + .00392 = .50392$.

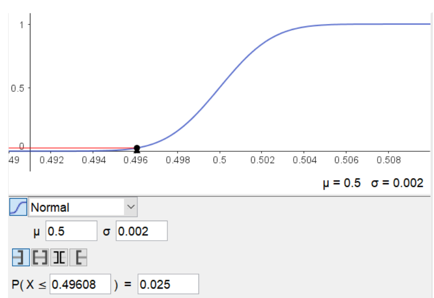
We can also find these values using GeoGebra or the calculator.

The figures and probability expressions from GeoGebra below show the two ways you can do this. If the first pair of figures I used the area under the pdf twice. First I entered .025 (X below) and then I entered .025 (X above). In the second pair of figures I used the cdf twice. First I set it equal to .025 and then I set it equal to .975. Both methods give $a = .49608$ and $b = .50392$.









The approach of the second set of figure is easiest to implement with the calculator. There

$$a = \text{ invNorm}(.975, .500, .002) = .50392 \text{ and } \text{invNorm}(.025, .500, .002) = .49608$$

3. A machine was designed to fill 12 ounce cups with a beverage. The target fill amount is exactly 12 oz and the acceptable fill range is 11.75 to 12.25 ounces. The amount of beverage dispensed by this machine can be reasonably modeled using a normal distribution with mean $\mu = 12$ oz and standard deviation $\sigma = .13$ oz.

a) Find the probability that the amount of beverage dispensed by this machine will be within the specified acceptable fill range.

b) If the machine dispenses more than 12.35 oz, then the cup will overflow wasting beverage and contaminating the outside of the cup. Find the probability that this machine will overfill the cup.

c) If the fill level in the cup is below 11.75 ounces, the customer will perceive the cup as being underfilled. Find the probability that the amount of beverage dispensed by the machine will be low enough to be perceived as underfilled by such a customer.

d) Using .95 (95%) as a "large proportion of the time", find the range of fill values produced using this machine with the property that a value in this range will occur with probability .95. That is, the range of values with the property that 95% of time the amount of beverage dispensed by this machine will be within this range.

Solution

Let X denote the amount of beverage dispensed by this machine.

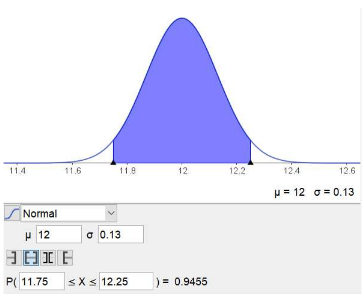
In terms of X we want to find $P(11.75 \leq X \leq 12.25)$. Using the calculator this is

$$P(11.75 \leq X \leq 12.25) = \text{normalcdf}(11.75, 12.25, 12, .13) = .9455$$

If we standardize we have

$$\begin{aligned} P(11.75 \leq X \leq 12.25) &= P\left(\frac{11.75 - 12}{.13} \leq Z \leq \frac{12.25 - 12}{.13}\right) \\ &= \text{normalcdf}\left(\frac{-.25}{.13}, \frac{.25}{.13}, 0, 1\right) \\ &= \text{normalcdf}\left(-1.9231, 1.9231, 0, 1\right) = .9455 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.



b) If the machine dispenses more than 12.35 oz, then the cup will overflow wasting beverage and contaminating the outside of the cup. Find the probability that this machine will overfill the cup.

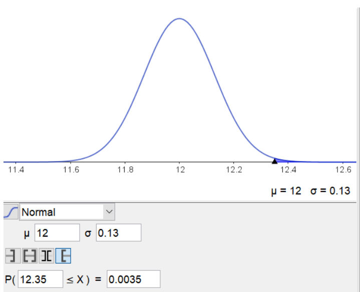
In terms of X we want to find $P(X \geq 12.35)$. Using the calculator this is

$$P(X \geq 12.35) = \text{normalcdf}(12.35, 1E99, 12, .13) = .0035$$

If we standardize we have

$$\begin{aligned} P(X \geq 12.35) &= P\left(X \geq \frac{12.35 - 12}{.13}\right) \\ &= \text{normalcdf}\left(\frac{.35}{.13}, 1E99, 0, 1\right) \\ &= \text{normalcdf}\left(2.6923, 1E99, 0, 1\right) = .0035 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.



c) If the fill level in the cup is below 11.75 ounces, the customer will perceive the cup as being underfilled. Find the probability that the amount of beverage dispensed by the machine will be low enough to be perceived as underfilled by such a customer.

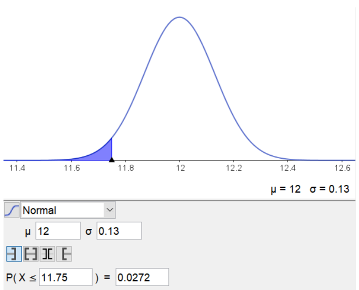
In terms of X we want to find $P(X \leq 11.75)$. Using the calculator this is

$$P(X \leq 11.75) = \text{normalcdf}(-1E99, 11.75, 12, .13) = .0272$$

If we standardize we have

$$\begin{aligned} P(X \leq 11.75) &= P\left(X \leq \frac{11.75 - 12}{.13}\right) \\ &= \text{normalcdf}\left(-1E99, \frac{-.25}{.13}, 0, 1\right) \\ &= \text{normalcdf}\left(-1E99, -1.9231, 0, 1\right) = .0272 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.



d) Using .95 (95%) as a “large proportion of the time”, find the range of fill values produced using this machine with the property that a value in this range will occur with probability .95. That is, the range of values with the property that 95% of time the amount of beverage dispensed by this machine will be within this range.

In terms of X we want to find a and b such that $P(a \leq X \leq b) = .95$. We can do this in two parts by finding the 2.5^{th} percentile and the 97.5^{th} percentile of the distribution of X. That is, we need to find a such that $P(X \leq a) = .025$ and find b such that $P(X \leq b) = .975$.

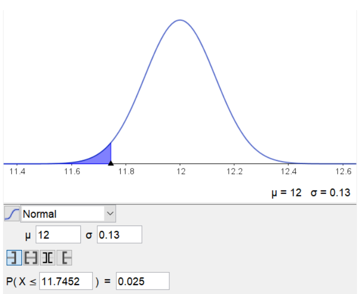
Using the calculator we have

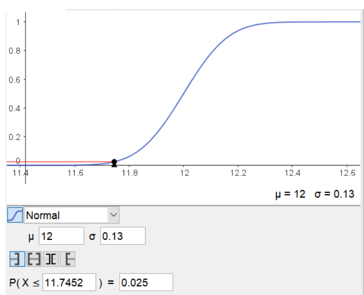
$$a = \text{invNorm}(.025, 12, .13) = 11.7452 \quad \text{ and } \quad b = \text{invNorm}(.975, 12, .13) = 12.2548$$

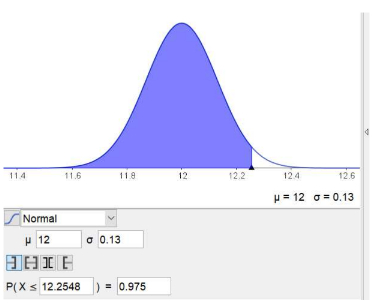
If we standardize, we can use the fact that $P(-1.96 \leq Z \leq 1.96) = .95$ giving

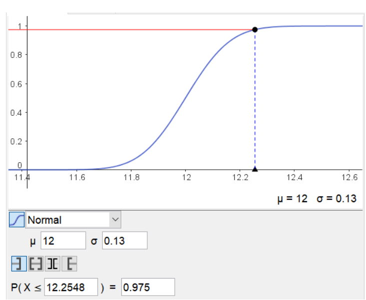
$$a = 12 - 1.96 \cdot .13 = 11.7452 \quad \text{ and } \quad b = 12 + 1.96 \cdot .13 = 12.2548$$

The figures and values of a and b from GeoGebra are shown in the screen captures below.

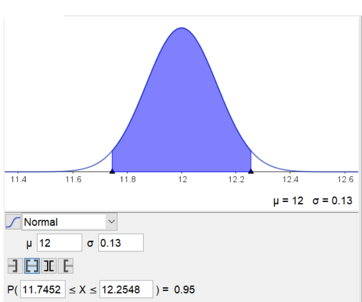








Let’s check our answer.



4. The blood cholesterol levels of middle aged men in the US can be reasonably modeled using a normal distribution with mean $\mu = 222$ mg/dl and standard deviation $\sigma = 36$ mg/dl.

a) Find the probability that the blood cholesterol level of a randomly selected man of this type would be considered high in the sense of exceeding 240 mg/dl.

b) Find the probability that the blood cholesterol level of a randomly selected man of this type would be considered elevated in the sense of falling in the range from 200 mg/dl to 240 mg/dl.

c) Find the probability that the blood cholesterol level of a randomly selected man of this type is either less than 186 mg/dl or greater than 312 mg/dl.

d) How would you characterize the largest 10% of all such blood cholesterol levels?

e) How would you characterize the smallest 5% of all such blood cholesterol levels?

Solution

Let X denote the blood cholesterol level of a middle aged man in the US (measured in mg/dl).

a) Find the probability that the blood cholesterol level of a randomly selected man of this type would be considered high in the sense of exceeding 240 mg/dl.

Let X denote the blood cholesterol level of one of these middle aged men.

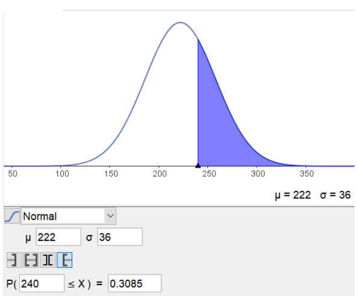
In terms of X we want to find $P(X \geq 240)$. Using the calculator this is

$$P(X \geq 240) = \text{normalcdf}(240, 1E99, 222, 36) = .3085$$

If we standardize we have

$$\begin{aligned} P(X \geq 240) &= P\left(Z \geq \frac{240-222}{36}\right) \\ &= \text{normalcdf}\left(-1E99, \frac{18}{36}, 0, 1\right) \\ &= \text{normalcdf}\left(-1E99, .5, 0, 1\right) = .3085 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.



b) Find the probability that the blood cholesterol level of a randomly selected man of this type would be considered elevated in the sense of falling in the range from 200 mg/dl to 240 mg/dl.

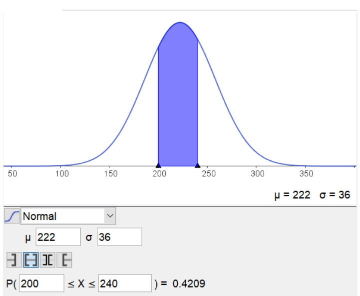
In terms of X we want to find $P(200 \leq X \leq 240)$. Using the calculator this is

$$P(200 \leq X \leq 240) = \text{normalcdf}(200, 240, 222, 36) = .4209$$

If we standardize we have

$$\begin{aligned} P(200 \leq X \leq 240) &= P\left(\frac{200-222}{36} \leq Z \leq \frac{240-222}{36}\right) \\ &= \text{normalcdf}\left(\frac{-22}{36}, \frac{18}{36}, 0, 1\right) \\ &= \text{normalcdf}\left(-.6111, .5, 0, 1\right) = .4209 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.



c) Find the probability that the blood cholesterol level of a randomly selected man of this type is either less than 186 mg/dl or greater than 312 mg/dl.

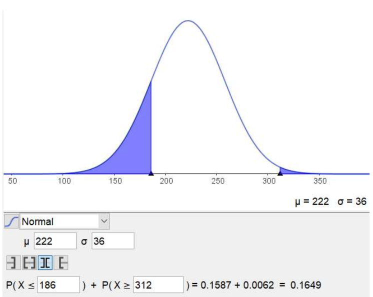
In terms of X we want to find $P(X \leq 186 \text{ or } X \geq 312)$. Using the calculator and the complement of this event we have

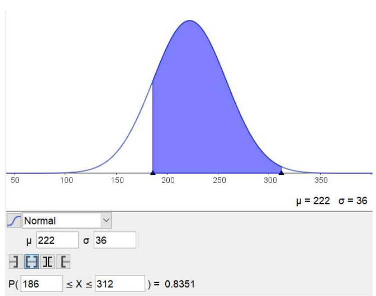
$$\begin{aligned} P(X \leq 186 \text{ or } X \geq 312) &= 1 - P(186 \leq X \leq 312) \\ &= 1 - \text{normalcdf}(186, 312, 222, 36) = 1 - .8351 = .1649 \end{aligned}$$

If we standardize we have

$$\begin{aligned} 1 - P(186 \leq X \leq 312) &= 1 - P\left(\frac{186-222}{36} \leq Z \leq \frac{312-222}{36}\right) \\ &= 1 - \ \text{normalcdf}\left(\frac{-36}{36}, \frac{90}{36}, 0, 1\right) \\ &= 1 - \ \text{normalcdf}\,(-1, 2.5, 0, 1) = 1 - .8351 = .1649 \end{aligned}$$

The figure and probability from GeoGebra are shown in the screen capture below.





d) How would you characterize the largest 10% of all such blood cholesterol levels?

In terms of X we want to find a such that $P(X \geq a) = .10$. We can do this by finding the 90^{th} percentile of the distribution of X. That is, we need to find a such that $P(X \leq a) = .90$.

Using the calculator we have

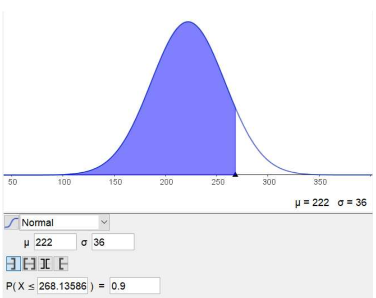
$$a = \text{invNorm}(.90, 222, 36) = 268.1359$$

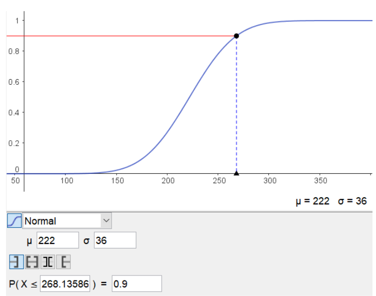
If we standardize, we need to find the 90^{th} percentile of the standard normal distribution. That is, find b such that $P(Z \leq b) = .90$. Doing this we have

$$b = \text{invNorm}(.90, 0, 1) = 1.281551567 \text{ therefore } a = 222 + 1.281551567 \cdot 36 = 268.1359$$

(If you use this method do not round until the very end.)

The figures and value of a from GeoGebra are shown in the screen captures below.





e) How would you characterize the smallest 5% of all such blood cholesterol levels?

In terms of X we want to find a such that $P(X \leq a) = .05$. We can do this by finding the 5^{th} percentile of the distribution of X. That is, we need to find a such that $P(X \leq a) = .05$.

Using the calculator we have

$$a = \text{invNorm}(.05, 222, 36) = 162.7853$$

If we standardize, we need to find the 5^{th} percentile of the standard normal distribution. That is, find b such that $P(Z \leq b) = .05$. Doing this we have

$$b = \text{invNorm}(.50, 0, 1) = -1.644853626 \text{ therefore } a = 222 - 1.644853626 \cdot 36 = 162.7853$$

(If you use this method do not round until the very end.) If, as is often done, we use the value $b = 1.645$ as the 5^{th} percentile of the standard normal distribution, then we get $a = 222 - 1.645 \cdot 36 = 162.78$

The figures and value of a from GeoGebra are shown in the screen captures below.