BUILDING HOMOTHETIES THROUGH PINCHES

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Abstract. We show that starting from finitely many points in $\mathbb{R}^m$, and successively applying "pinches" on them, it is possible to arrive at any of the configurations that result from applying, to the original points, a homothety of factor $0 \leqslant s \leqslant 1$ and center the centroid of the points. Here, a "pinch" consists in moving two points towards their common centroid. For three points, four pinches suffice. In general, the number of pinches is independent of the original configuration and of m.

1. The problem

We start with a finite collection of points $P_1, P_2, \ldots, P_n$ in $\mathbb{R}^m$. We allow ourselves one specific "move" that alters their configuration: we can take any two points and scale them towards their common centroid by any factor in $[0,1]$. The goal is to scale the whole collection of points by a prescribed factor $0 \leqslant s \leqslant 1$ towards their centroid in a finite number of moves.

The points have weights associated to them, which are used to determine the centroid of any pair of points, and of the whole collection. Let $w_k > 0$ be the weight of P_k for $k = 1, \ldots, n$. Applying the basic move of the problem to points P_i and P_j, with weights w_i and w_j, results in P_i' and P_j', where P_i' and P_j' are obtained from P_i and P_j by a homothety that has their centroid $(w_iP_i + w_jP_j)/(w_i + w_j)$ as its center and any number in $[0,1]$ as its scaling factor. We refer to this operation as a *pinch*. The problem formulated above asks whether the configuration obtained by applying a homothety to all the points, with a factor $0 \leqslant s \leqslant 1$ and with center the centroid of the points, is attainable via a finite series of pinches. We solve this problem affirmatively in the coming sections (Theorem 9). The special case of the homothety with factor $s = 0$ (that collapses all points to the centroid) has been remarked upon before, assuming also equal weights; see [1, Theorem 1.2]. But allowing for an arbitrary $0 \leqslant s \leqslant 1$, and the introduction of weights, do complicate matters considerably.

It is difficult to describe exactly which configurations of points are attainable from a given initial configuration by repeated applications of pinches. It follows from our main result, however, that the attainable configurations form a star shaped set (Corollary 10). Since pinches do not change neither the centroid of the points, nor the affine space that they generate, the attainable configurations must have the same centroid as the initial one, and be contained in its affine span, but these are only two of many limitations. In the case that the points have equal weights, the attainable

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configurations are *majorized* by the initial one, in the sense of [2]. This is, however, not sufficient. We elaborate on the relation of the problem to majorization, and to the closely related notion of chain majorization, in Section 3.

2. Formalization of the problem

Let $\mathcal{P}_{n,m}$ denote the vector space of n-tuples of points $(P_1, P_2, \ldots, P_n)$, with $P_k \in \mathbb{R}^m$ for all k. We write the n-tuple $(P_1, P_2, \ldots, P_n)$ as an $n \times m$ matrix with rows $P_1, P_2, \ldots, P_n$. Geometrically, we think of elements in $\mathcal{P}_{n,m}$ as collections of n labeled points in $\mathbb{R}^m$.

Let us fix weights $w_1, w_2, \ldots, w_n$, with $w_k > 0$ for all k. We regard the k-th point P_k of an n-tuple $(P_1, P_2, \ldots, P_n)$ in $\mathcal{P}_{n,m}$ as having weight w_k, for all k.

Given two distinct indices $1 \leqslant i, j \leqslant n$ and a real number s, we call a *2-homothety with parameter s acting on P_i and P_j* the linear transformation

$$(P_1, P_2, \ldots, P_n) \mapsto (P_1', P_2', \ldots, P_n'),$$

from $\mathcal{P}_{n,m}$ to itself such that

$$P_k' = \begin{cases} sP_k + (1-s)\frac{w_iP_i + w_jP_j}{w_i + w_j} & \text{if } k = i, j, \\ P_k & \text{otherwise.} \end{cases}$$

This transformation applies a homothety to P_i and P_j, with scaling factor s and center the centroid of P_i and P_j, while leaving the other points unchanged. We denote by $T_{P_iP_j}(s)$ the 2-homothety on P_i and P_j with scaling factor s. If $0 \leqslant s \leqslant 1$, we call such a 2-homothety a pinch.

We call an n-homothety, or total homothety, the transformation on $\mathcal{P}_{n,m}$ that applies a homothety to all the points of an n-tuple with center the centroid of these points. More specifically, the total homothety on $\mathcal{P}_{n,m}$ with scaling factor s is the linear transformation

$$(P_1, P_2, \ldots, P_n) \mapsto (P_1', P_2', \ldots, P_n'),$$

such that

$$P_k' = sP_k + (1-s)\frac{w_1P_1 + w_2P_2 + \cdots + w_nP_n}{w_1 + w_2 + \cdots + w_n},$$

for all k. We denote the total homothety with factor s on $\mathcal{P}_{n,m}$ by $H(s)$.

The problem from the introduction can now be restated as follows:

Problem 1. *Is the total homothety with scaling factor $0 \leqslant s \leqslant 1$ on the vector space $\mathcal{P}_{n,m}$ expressible as a composition of pinches, i.e., of 2-homotheties with parameters in $[0,1]$?*

Let i, j be distinct indices between 1 and n. Let $E^{i,j}$ denote the $n \times n$ row stochastic matrix whose entries in the (i,i) and (j,i) positions equal $\frac{w_i}{w_i+w_j}$, whose entries in the (i,j), and (j,j) positions equal to $\frac{w_j}{w_i+w_j}$, and otherwise whose off diagonal entries are

0 and diagonal entries are 1. For example, if $n = 4$, $i = 1$, and $j = 2$, then

$$E^{1,2} = \begin{pmatrix} \frac{w_1}{w_1+w_2} & \frac{w_2}{w_1+w_2} & 0 & 0 \\ \frac{w_1}{w_1+w_2} & \frac{w_2}{w_1+w_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now let $s \in \mathbb{R}$ and define

$$T_s^{i,j} = sI_n + (1-s)E^{i,j},$$

where I_n is the $n \times n$ identity matrix. Let $P = (P_1, \ldots, P_n)$ be an n-tuple in $\mathcal{P}_{n,m}$. A straightfoward calculation then shows that $T_s^{i,j}P$ is the result of applying the 2-homothety on P_i, P_j to the n-tuple P, i.e,

$$T_{P_iP_j}(s)(P) = T_s^{i,j}P.$$

If $0 \leqslant s \leqslant 1$, then we call $T_s^{i,j}$ a pinching matrix. (Note: This differs from the terminology found in [2] and in other instances in the literature, where pinching matrices have parameter $-1 \leqslant s \leqslant 1$.)

Let E be the $n \times n$ matrix whose (i,j) entry is equal to $\frac{w_j}{w}$ for all $1 \leqslant i, j \leqslant n$, where $w = \sum_{k=1}^{n} w_k$. Now let $s \in \mathbb{R}$ and define

$$H_s = sI_n + (1-s)E.$$

Again, a striaghtforward calculation shows that H_sP is the result of applying the total homothety with parameter s on P, i.e., $H(s)(P) = H_sP$.

Problem 1 can be restated as asking whether for any $P \in \mathcal{P}_{n,m}$ and $0 \leqslant s \leqslant 1$ there exist $n \times n$ pinching matrices $T_1, \ldots, T_N$ such that

$$T_N T_{N-1} \cdots T_1 P = H_s P.$$

If we let $m = n$, and we choose our n-tuple of points $P \in \mathcal{P}_{n,n}$ so that $P = I_n$, then we are asking to show that

$$T_N T_{N-1} \cdots T_1 = H_s.$$

Clearly, proving this formula is equivalent to solving the original problem for all n and m. Consequently, we can reformulate the problem as asking whether the matrices of the form H_s, with $0 \leqslant s \leqslant 1$, are expressible as products of pinching matrices.

3. Relation to multivariate majorization.

Let us briefly discuss the problem of characterizing which n-tuples in $\mathcal{P}_{n,m}$ are attainable from a starting n-tuple via pinches. For simplicity, we assume equal weights, i.e., $w_1 = w_2 = \cdots = w_n$.

Let P and Q be in $\mathcal{P}_{n,m}$. In light of our discussion in Section 2, Q is attainable from P via pinches if and only if

$$Q = (T_N T_{N-1} \cdots T_1)P,$$

where the T_is are $n \times n$ pinching matrices. From the equal weights assumption, we also have that every pinching matrix $T_s^{i,j}$, with $0 \leqslant s \leqslant 1$, is doubly stochastic, i.e., it has nonnegative entries and both its rows and columns add up to 1. Since the product of doubly stochastic matrices is again doubly stochastic, it follows that a necessary condition for Q to be attainable from P via pinches is that $Q = DP$, where D is an $n \times n$ doubly stochastic matrix. The relation $Q = DP$ for a doubly stochastic D is called *multivariate majorization* (see [2, Chapter 15]).

It is in general not sufficient that P majorizes Q for Q to be attainable from P via pinches. Let us consider n-tuples in $\mathcal{P}_{n,1}$, i.e., vectors in $\mathbb{R}^n$. In this case the situation is well understood. A classical result in the theory of majorization states that a vector $P \in \mathbb{R}^n$ majorizes another one $Q \in \mathbb{R}^n$ if and only if Q is attainable from P via 2-homotheties with parameter $-1 \leqslant s \leqslant 1$ (these are also called tranfers or T-transforms in the literature; see [2, Chapter 2, Lemma B.1]). It is not difficult to deduce from this that if Q is majorized by P, then either Q or a permutation of the points in Q is attainable from P through pinches (2-homotheties with parameter $0 \leqslant s \leqslant 1$). The question of exactly which vectors $Q \in \mathbb{R}^n$ are attainable via pinches from a given P is more nuanced, but has also been solved by Zylka in [3].

For $m > 1$, the limitations on which Q majorized by P can be attained via pinches are even stricter, and no general characterization is known. We illustrate this with a well known example (see [2, Chapter 15, Example A.3]). Consider a nondegenerate triangle $P = (A, B, C)$ in $\mathcal{P}_{3,2}$ (i.e., three non-collinear points on the plane) and the triangle formed by the midpoints of its sides $Q = (A', B', C')$. See Figure 1. In this case, Q is majorized by P, but any non-trivial pinch on P produces a triangle that does not contain Q. Hence, neither Q nor a permutation of its points is attainable from P via pinches.

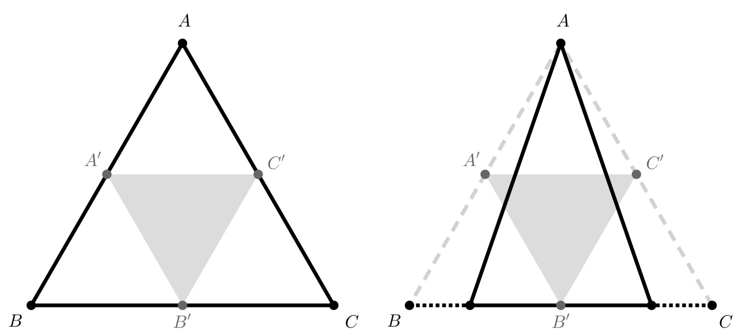

FIGURE 1. On the left: the triangle $P = (A, B, C)$ and, in gray, the triangle $Q = (A', B', C')$ formed by the midpoints of its sides. On the right: the triangle resulting from a pinch performed on one of P's sides.

The relation "Q is attainable from P via 2-homotheties with parameter in $[-1, 1]$", termed *chain majorization* in [2], is also in general difficult to understand. Zylka's above mentioned result in [3] shows that, already for $m = 1$, chain majorization is a strictly weaker relation than the relation that occupies us here, i.e, attainability via pinches.

4. The case of three points.

In this section, we solve Problem 1 for three points in $\mathbb{R}^m$.

Let us denote the three points on which we operate by A, B, C, and their weights by w_A, w_B, w_C, respectively. We denote a generic triple in the vector space $\mathcal{P}_{3,m}$ by (A, B, C). Accordingly, we denote by $T_{AB}(x)$, $T_{BC}(x)$ and $T_{AC}(x)$ the 2-homotheties (with parameter x) between these points.

Let $x \in \mathbb{R}$. Define numbers $s, y, z \in \mathbb{R}$ by the formulas

$$s = \frac{x(wx - w_C)}{w - w_C x}, \; y = \frac{(w_A x + w_B)(wx - w_C)}{(w_B x + w_A)(w - w_C x)}, \; z = \frac{(w_B x + w_A)(wx - w_C)}{(w_A x + w_B)(w - w_C x)}. \tag{4.1}$$

Here, $w = w_A + w_B + w_C$ and we assume that $x \notin \{\frac{w}{w_C}, -\frac{w_B}{w_A}, -\frac{w_A}{w_B}\}$.

Lemma 2. *We have that*

$$H(s) = T_{BC}(z) T_{AB}(x) T_{AC}(y) T_{AB}(x),$$

where the left hand side is the total homothety on $\mathcal{P}_{3,m}$, the right hand side is a product of 2-homotheties in $\mathcal{P}_{3,m}$, and s, y, and z are expressed in terms of x according to the formulas (4.1).

Proof. As discussed in Section 2, we can represent 2-homotheties and total homotheties by matrix multiplications. The lemma then boils down to proving the identity

$$H_s = T_z^{BC} T_x^{AB} T_y^{AC} T_x^{AB}, \tag{4.2}$$

where T_x^{AB}, T_y^{AC}, T_z^{BC}, and H_s are the matrices associated to $T_{AB}(x)$, $T_{AC}(y)$, $T_{BC}(z)$, and $H(s)$, respectively. We have that

$$T_x^{AB} = xI_3 + (1-x)E^{AB} = \begin{pmatrix} \frac{w_B x + w_A}{w_A + w_B} & \frac{w_B(1-x)}{w_A + w_B} & 0 \\ \frac{w_B(1-x)}{w_A + w_B} & \frac{w_A x + w_B}{w_A + w_B} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly

$$T_y^{AC} = \begin{pmatrix} \frac{w_C y + w_A}{w_A + w_C} & 0 & \frac{w_C(1-y)}{w_A + w_C} \\ 0 & 1 & 0 \\ \frac{w_C(1-y)}{w_A + w_C} & 0 & \frac{w_A y + w_C}{w_A + w_C} \end{pmatrix}, \qquad T_z^{BC} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{w_C z + w_B}{w_B + w_C} & \frac{w_C(1-z)}{w_B + w_C} \\ 0 & \frac{w_C(1-z)}{w_B + w_C} & \frac{w_B z + w_C}{w_B + w_C} \end{pmatrix},$$

$$H_s = sI_3 + (1-s)\begin{pmatrix} \frac{w_A}{w} & \frac{w_B}{w} & \frac{w_C}{w} \\ \frac{w_A}{w} & \frac{w_B}{w} & \frac{w_C}{w} \\ \frac{w_A}{w} & \frac{w_B}{w} & \frac{w_C}{w} \end{pmatrix}.$$

Checking (4.2) is a straightforward, though lengthy calculation that can be carried out using a symbolic calculations software (we used Python with SymPy). $\square$

Lemma 3. *Let $0 \leqslant s \leqslant 1$. There then exists a unique $\frac{w_C}{w} \leqslant x \leqslant 1$ solving the equation*

$$s = \frac{x(wx - w_C)}{w - w_C x}.$$

Moreover, in this case the numbers y and z given by the formulas (4.1) *end up in the interval* $[0, 1]$.

Proof. Let us regard s, y, z as functions of x through the formulas (4.1). For $x = \frac{w_C}{w}$ we have $s, y, z = 0$, and for $x = 1$ we have $s, y, z = 1$. To complete the proof of the lemma, it will suffice to show that s, y, z are strictly increasing functions of x on the interval $\frac{w_C}{w} < x < 1$. This can be done by taking logarithmic derivatives of their formulas with respect to x, and checking that these are greater than zero.

$$\begin{aligned} \frac{s'}{s} &= \frac{1}{x} + \frac{w}{wx - w_C} + \frac{w_C}{w - w_C x}, \\ \frac{y'}{y} &= \frac{w_A}{w_A x + w_B} + \frac{w}{wx - w_C} + \frac{w_C}{w - w_C x} - \frac{w_B}{w_B x + w_A}. \end{aligned}$$

We omit the logarithmic derivative of z, as the cases of z and y are the same by symmetry. It is clear from the above formula that $\frac{s'}{s} > 0$ for $\frac{w_C}{w} < x < 1$. For $\frac{y'}{y}$ we notice first that $\frac{w}{wx - w_C} \geqslant \frac{w_B}{w_B x + w_A}$, from which we again deduce that $\frac{y'}{y} > 0$. ☐

Theorem 4. *For every $0 \leqslant s \leqslant 1$, the total homothety $H(s)$ on $\mathcal{P}_{3,m}$ is a product of four pinches.*

Proof. By Lemma 2, $H(s)$ is a product of four 2-homotheties, which by Lemma 3 are pinches. ☐

Let us explicitly describe a sequence of steps that solve the problem from the introduction. Let $0 \leqslant s \leqslant 1$. Let x be the unique solution of $s = \frac{x(wx - w_C)}{w - w_C x}$ such that $\frac{w_C}{w} \leqslant x \leqslant 1$. To construct the total homothety on A, B, C with parameter s, we may proceed as follows:

(1) pinch A and B by a factor of x,
(2) pinch A and C by a factor of

$$y = \frac{w_A x + w_B}{w_B x + w_A} \cdot \frac{s}{x},$$

(3) pinch A and B by a factor of x again,
(4) pinch B and C by a factor of

$$z = \frac{w_B x + w_A}{w_A x + w_B} \cdot \frac{s}{x}.$$

As demonstrated in Lemma 3, these steps are indeed pinches, i.e., the scaling factors x, y, z are all in the interval $[0, 1]$. Further, by Lemma 2, after these four pinches, A, B, C are situated in their target positions, i.e., the triangle A, B, C has been scaled towards its centroid (determined using the weights w_A, w_B, w_C) by a factor of s. The process is illustrated in Figure 2.

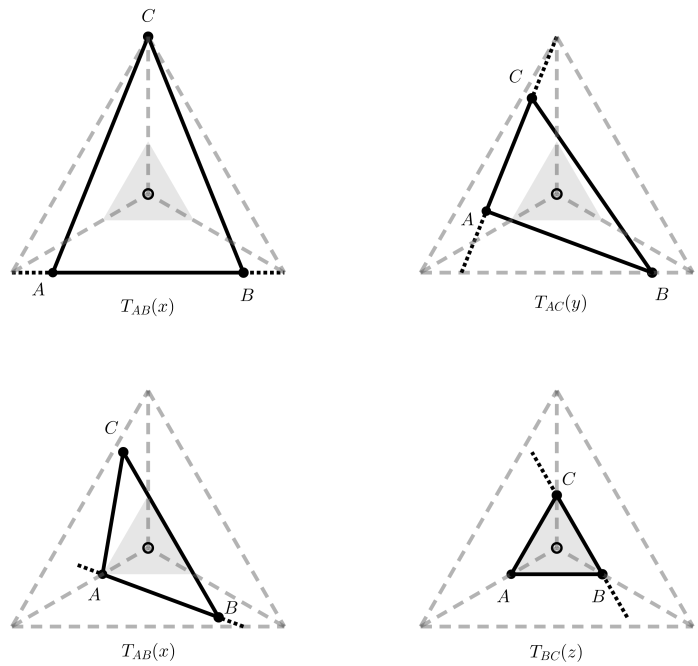

FIGURE 2. Four pinches applied to A, B, C to obtain a homothety (with the center denoted by a hollow circle). The target triangle is pictured in gray, and the original triangle and its medians are pictured with gray, dashed lines. All points are given equal weights for simplification.

The following observation plays a crucial role in the solution of the general case: Applying the first three steps of the procedure above has the effect that both A and the centroid of B and C are already in their target positions, since the last step—a pinch on B and C—does not change the centroid of B, C. Thus, these three steps have the same effect on A and the centroid of B and C, call it M, as a pinch on A and M with parameter s, where the weight of M is $w_B + w_C$.

Let us express this observation more formally. Denote an arbitrary pair in $\mathcal{P}_{2,m}$ by (A, M). Associate to the pairs in this space the weights (w_A, w_M), where $w_M = w_A + w_B$. Denote the 2-homothety on $\mathcal{P}_{2,m}$ with parameter s by $T_{AM}(s)$.

Define $\Phi\colon \mathcal{P}_{3,m} \to \mathcal{P}_{2,m}$ as

$$\Phi(A, B, C) = \Big(A, \frac{w_B}{w_M}B + \frac{w_C}{w_M}C\Big).$$

That is, Φ sends a triple of points (A, B, C) to the pair consisting of A and the centroid of B and C. Observe that, according to the way that we have defined the weights in $\mathcal{P}_{2,m}$, the centroids of (A, B, C) and $\Phi(A, B, C)$ agree.

Let $K\colon \mathcal{P}_{3,m} \to \mathcal{P}_{3,m}$ denote the transformation defined by

$$K = T_{AB}(x)T_{AC}(y)T_{AB}(x), \tag{4.3}$$

where s, x, y are related by the formulas (4.1). That is, K consists of the first three 2-homotheties in the formula from Lemma 2.

Lemma 5. *We have that* $T_{AM}(s)\Phi = \Phi K$.

Proof. By Lemma 2, we have that $H(s) = T_{BC}(z)K$. Composing both sides by Φ we get $\Phi H(s) = \Phi T_{BC} K$. We readily verify that $\Phi H(s) = T_{AM}(s)\Phi$. On the other hand, $\Phi T_{BC}(z) = \Phi$, since a 2-homothety between B and C does not change their centroid. Hence, $T_{AM}(s)\Phi = \Phi K$, as desired. $\square$

Yet another observation that we will need below is contained in the following lemma:

Lemma 6. *Let* $(A_0, B_0, C_0) \in \mathcal{P}_{3,m}$ *and let* $(A_1, B_1, C_1) = K(A_0, B_0, C_0)$*, where* K *is the map defined in* (4.3)*. Then,*

$$B_1 - C_1 = \frac{x(w_A x + w_B)}{w_B x + w_A}(B_0 - C_0).$$

Proof. Let us first assume that $x > \frac{w_C}{w}$, so that $z > 0$. Set

$$(A_2, B_2, C_2) = H(s)(A_0, B_0, C_0).$$

Then, $B_2 - C_2 = s(B_0 - C_0)$. On the other hand, since

$$T_{BC}(z)(A_1, B_1, C_1) = (A_2, B_2, C_2),$$

we have that $B_2 - C_2 = z(B_1 - C_1)$. Hence,

$$B_1 - C_1 = \frac{s}{z}(B_0 - C_0) = \frac{x(w_A x + w_B)}{w_B x + w_A}(B_0 - C_0).$$

The case $x = \frac{w_C}{w}$ now follows by the continuity of B_1 and C_1 with respect to x. $\square$

5. The general case.

In this section we solve Problem 1 from the introduction for the general case of n points in $\mathbb{R}^m$ with weights $w_1, w_2, \dots, w_n$. We first derive a formula that expresses the total homothety on $\mathcal{P}_{n,m}$ as a product of 2-homotheties. Then, we address the question of turning these 2-homotheties into pinches (i.e., with a scaling factor in the range $[0, 1]$).

We will obtain the solution for n points recursively, starting with the solution for three points, and working our way up to n points. Throughout this section we assume that $n \geqslant 3$.

Let us introduce the function

$$F(s,t) = \frac{t(1-s) + \sqrt{t^2(1-s^2) + 4s}}{2}, \tag{5.1}$$

where $s \geqslant 0$ and $t > 0$. Observe that $x = F(s,t)$ is the unique positive solution of the equation

$$s = \frac{x(x-t)}{1-tx}.$$

(Thus, the number x from Lemma 3 is simply $x = F(s, \frac{w_C}{w})$.)

Let us now produce recursive formulas for numbers

$$x_{k,i}, \text{ with } 1 \leqslant i < k \leqslant n-1,$$
$$y_{k,i}, \text{ with } 1 \leqslant i < k \leqslant n,$$

as follows: Define $y_{2,1} = s$, and then letting k range through $2, 3, \ldots, n-1$, define

$$(5.2)\qquad \begin{aligned} &\left.\begin{aligned} x_{k,i} &= F\Big(y_{k,i}, \frac{w_{k+1} + \cdots + w_n}{w_i + w_k + w_{k+1} + \cdots + w_n}\Big) \\ y_{k+1,i} &= \frac{w_i x_{k,i} + w_k}{w_k x_{k,i} + w_i} \cdot \frac{y_{k,i}}{x_{k,i}} \end{aligned}\right\} \text{ for } i = 1, 2 \ldots, k-1, \\ &y_{k+1,k} = \frac{s}{\prod_{i=1}^{k-1} x_{k,i} \cdot \frac{w_i x_{k,i} + w_k}{w_k x_{k,i} + w_i}}. \end{aligned}$$

Theorem 7. *Let $0 \leqslant s \leqslant 1$. The total homothety $H(s)$ on $\mathcal{P}_{n,m}$ is equal to the product of $(n-1)^2$ 2-homotheties obtained as follows: For each $i = 1, 2, \ldots, n-2$, form the product*

$$S_i = \Big(\prod_{j=i+1}^{n-1} T_{P_{n+i-j}P_i}(x_{n+i-j,i})\Big) \cdot T_{P_nP_i}(y_{n,i}) \cdot \Big(\prod_{j=i+1}^{n-1} T_{P_jP_i}(x_{j,i})\Big),$$

and set $S_{n-1} = T_{P_nP_{n-1}}(y_{n,n-1})$. Then

$$H(s) = S_{n-1}S_{n-2}\cdots S_1.$$

The above formula for $H(s)$ can be understood by arranging the numbers $x_{k,i}$, for $1 \leqslant i < k \leqslant n-1$, and $y_{i,n}$, for $i = 1, 2 \ldots, n-1$, in a triangular array as follows:

$$\begin{array}{cccccc} & P_1 & P_2 & \ldots & P_{n-2} & P_{n-1} \\ P_2 & x_{2,1} & & & & \\ P_3 & x_{3,1} & x_{3,2} & & & \\ \vdots & \vdots & \vdots & \vdots & & \\ P_{n-1} & x_{n-1,1} & \ldots & \ldots & x_{n-1,n-2} & \\ P_n & y_{n,1} & \ldots & \ldots & y_{n,n-2} & y_{n,n-1} \end{array}$$

The parameter at the crossing of any two points is the parameter of their 2-homothety. Furthermore, starting with the first column, by scanning each column down and back up, we add to the product each 2-homothety, where the factor $x_{k,i}$ is used on a 2-homothety between P_k and P_i, and similarly, the factor $y_{n,i}$ is used on a 2-homothety between P_n and P_i. Moreover, the first column lists the factors used in S_1, the second column the factors in S_2, and so on. Theorem 7 states that the result of this composition of 2-homotheties agrees with the total homothety $H(s)$.

Proof of Theorem 7. We proceed by induction on n. For $n = 3$, the formula is the same as the one obtained in Lemma 2. Let us assume, then, that $n > 3$, and that the formula is true in the space of $(n-1)$-tuples $\mathcal{P}_{n-1,m}$, for any collection of $n-1$ weights associated to this space.

Fix a collection of n weights $(w_1, w_2, \ldots, w_n)$, and consider the space of n-tuples $\mathcal{P}_{n,m}$ with these weights associated to them.

To the space of $(n-1)$-tuples $\mathcal{P}_{n-1,m}$, let us associate the weights

$$w_1, w_2, \ldots, w_{n-2}, w_M,$$

where $w_M = w_{n-1} + w_n$. Let us denote a generic $(n-1)$-tuple in $\mathcal{P}_{n-1,m}$ by $(P_1, P_2, \ldots, P_{n-2}, M)$. Finally, let $H_{n-1}(s)$ denote the total homothety in this space. By the induction hypothesis, $H_{n-1}(s)$ is a product of $(n-2)^2$ 2-homotheties. Examination of the recursive formulas for the factors of these 2-homotheties reveals that they are obtained as follows: We first form the triangular array

$$\begin{array}{cccccc}
 & P_1 & P_2 & \ldots & P_{n-3} & P_{n-2} \\
P_2 & x_{2,1} & & & & \\
P_3 & x_{3,1} & x_{3,2} & & & \\
\vdots & \vdots & \vdots & \vdots & & \\
P_{n-2} & x_{n-2,1} & \cdots & \cdots & x_{n-2,n-3} & \\
P_{n-1} & y_{n-1,1} & \cdots & \cdots & y_{n-1,n-3} & y_{n-1,n-2},
\end{array}$$

where the $x_{k,i}$s and $y_{k,i}$s are the same factors as before in the recursive formulas (5.2). Observe, however, that the last row consists of the factors $y_{n-1,i}$, which are defined through (5.2), but which do not appear in the formula that we wish to prove for the total homothety on $\mathcal{P}_{n,m}$. Now, define

$$S'_i = \Big(\prod_{j=i+1}^{n-2} T_{P_{n+i-j}P_i}(x_{n+i-j,i}) \Big) \cdot T_{MP_i}(y_{n-1,i}) \cdot \Big(\prod_{j=i+1}^{n-2} T_{P_jP_i}(x_{j,i}) \Big)$$

for $i = 1, 2, \ldots, n-3$, and $S'_{n-2} = T_{MP_{n-2}}(y_{n-1,n-2})$. The parameters in the 2-homotheties of the product S'_i are obtained by scanning down and back up along the i-th column of the triangular array above. Then, by the induction hypothesis,

$$H_{n-1}(s) = S'_{n-2} \cdots S'_1.$$

We now construct a product of 2-homotheties in $\mathcal{P}_{n,m}$ as follows: Scanning through the array for $H_{n-1}(s)$, the 2-homotheties between points P_i and P_j with $i, j \leqslant n-2$ are left unchanged, although now taken in the space $\mathcal{P}_{n,m}$. Each factor $T_{MP_i}(y_{n-1,i})$, on the other hand, is replaced with the product of the first three 2-homotheties in the formula (2) from the solution of the $n=3$ case, applied to the triangle $A = P_i$, $B = P_{n-1}$, $C = P_n$, and for the total homothety on that triangle with factor $y_{n-1,i}$. By the way that we have defined the recursive formulas (5.2), this product is

$$K_i = T_{P_{n-1}P_i}(x_{n-1,i}) T_{P_nP_i}(y_{n,i}) T_{P_{n-1}P_i}(x_{n-1,i}). \tag{5.3}$$

The substitution of $T_{MP_i}(y_{n-1,i})$ by K_i in S'_i results in the product S_i from the statement of the theorem. Thus, the new product of 2-homotheties obtained by this procedure consists precisely of the first $(n-1)^2 - 1$ terms of the formula for $H(s)$ that we wish to prove. Let us call this product K. That is, $K = S_{n-2}S_{n-1}\cdots S_1$, where S_i is as in the statement of the theorem.

Fix an n-tuple $P^{(0)} = (P_1^{(0)}, P_2^{(0)}, \dots, P_n^{(0)})$ in $\mathcal{P}_{n,m}$. Let us examine the effect of applying K to $P^{(0)}$. Set $P^{(1)} = K(P^{(0)})$ and $P^{(2)} = H(s)(P^{(0)})$. We claim that

(1) $P_i^{(1)} = P_i^{(2)}$ for $i = 1, 2\dots, n-2$, and
(2) the centroids of $P_{n-1}^{(1)}, P_n^{(1)}$ and $P_{n-1}^{(2)}, P_n^{(2)}$ agree.

Indeed, by Lemma 5, after substituting $T_{P_iM}(y_{n-1,i})$ by K_i we indirectly apply a 2-homothety with parameter $y_{n-1,i}$ on P_i and the centroid of P_{n-1} and P_n. Thus, the product of 2-homotheties that we have built has the same effect on the $n-1$-tuple $(P_1^{(0)}, P_2^{(0)}, \dots, P_{n-2}^{(0)}, M^{(0)})$ as directly applying the solution of the $n-1$ case to these points. More formally, let $\Phi\colon \mathcal{P}_{n,m} \to \mathcal{P}_{n-1,m}$ denote the transformation such that

$$(P_1, \dots, P_n) \overset{\Phi}{\mapsto} (P_1, \dots, P_{n-2}, \frac{w_{n-1}P_{n-1} + wnP_n}{w_{n-1} + w_n}).$$

Then, by Lemma 5,

$$T_{P_iM}(y_{n-1,i})\Phi = \Phi T_{P_jP_{n-1}}(x_{n-1,i})T_{P_iP_n}(y_{n,i})T_{P_iP_{n-1}}(x_{n-1,i}),$$

for all $i = 1, \dots, n-2$. From this we deduce that $\Phi K = H_{n-1}(s)\Phi$. The claims (1) and (2) made above readily follow.

Observe now that the transformation K_i, defined as in (5.3), has the effect of multiplying $P_n - P_{n-1}$ by the scalar

$$x_{n,i} \cdot \frac{w_i x_{n,i} + w_n}{w_n x_{n,i} + w_i}$$

(by Lemma 6). It follows that

$$P_n^{(1)} - P_{n-1}^{(1)} = z(P_n^{(0)} - P_{n-1}^{(0)}),$$

where

$$z = \prod_{i=1}^{n-2} x_{n,i}\frac{w_i x_{n,i} + w_n}{w_n x_{n,i} + w_i}.$$

From the recursive formulas (5.2) that define the $x_{k,i}$s and $y_{k,i}$s, we readily see that $y_{n,n-1} = \frac{s}{z}$. Thus, after applying $T_{P_nP_{n-1}}(y_{n,n-1})$ to $P^{(1)}$, we obtain an n-tuple

$$(P_1^{(2)}, \dots, P_{n-2}^{(2)}, P'_{n-1}, P'_n)$$

such that $P'_{n-1} - P'_n = P_{n-1}^{(2)} - P_n^{(2)}$, and the centroids of P'_{n-1}, P'_n and $P_{n-1}^{(2)}, P_n^{(2)}$ agree. This readily implies that $P'_{n-1} = P_{n-1}^{(2)}$ and $P'_n = P_n^{(2)}$. Hence,

$$T_{P_nP_{n-1}}(y_{n,n-1})K(P^{(0)}) = P^{(2)} = H(s)(P^{(0)}).$$

Since $P^{(0)}$ can be varied arbitrarily, this shows that $T_{P_nP_{n-1}}(y_{n,n-1})K = H(s)$, which completes the induction. $\square$

It remains to be shown that the 2-homotheties in Theorem 7 are in fact all pinches. We conjecture that this is indeed always true, although a proof has elluded us. However, for a complete solution of Problem 1, the following theorem is sufficient:

Theorem 8. *There exists an $\varepsilon > 0$ such that for $s \in [0,\varepsilon]$ and $s \in [1-\varepsilon, 1]$, the factors $x_{k,i}$ and $y_{n,i}$ in the formula for $H(s)$ of Theorem 7 are all in the interval $[0,1]$. Consequently, for this range of values of s the total homothety $H(s)$ on $\mathcal{P}_{n,m}$ is a product of $(n-1)^2$ pinches.*

Proof. Since $F(s,t) \geqslant 0$ for $s,t \geqslant 0$, it is clear from (5.2) that the $x_{k,i}$s and $y_{k,i}$s are all non-negative for $s \in [0,1]$. Next, we focus on bounding these numbers by 1 from above.

Let us first deal with s close to 0. We regard the $x_{k,i}$s and $y_{k,i}$s as functions of s. As such, they are clearly continuous. Notice then that setting $s = 0$ in (5.2), and using that $F(0,t) = t$, we obtain at once that $x_{k,i} = t_{k,i}$ and $y_{k,i} = 0$ for all k, i. Since $t_{k,i} < 1$, it follows from the continuity of the $x_{k,i}$s and $y_{k,i}$s with respect to s that, for s close enough to 0, they are all less than 1.

Next, let us deal with s close to 1. We continue to regard the $x_{k,i}$s and $y_{k,i}$s as functions of s. Setting $s = 1$ in (5.2), and using that $F(1,t) = 1$, we obtain at once that $x_{k,i}(1) = y_{k,i}(1) = 1$ for all k, i. To fulfill our goal, it is sufficient to show that $\frac{d}{ds}x_{k,i}|_{s=1} > 0$ and $\frac{d}{ds}y_{k,i}|_{s=1} > 0$. We prove this next.

Set

$$a_{k,i} = \frac{d}{ds}x_{k,i}\Big|_{s=1}, \text{ for } 1 \leqslant i < k \leqslant n-1,$$
$$b_{k,i} = \frac{d}{ds}y_{k,i}\Big|_{s=1}, \text{ for } 1 \leqslant i < k \leqslant n.$$

We shall derive recursive formulas for these numbers.

From the initial condition $y_{2,1} = s$ we deduce at once that

$$b_{2,1} = \frac{d}{ds}y_{2,1}\Big|_{s=1} = 1. \tag{5.4}$$

Recall that

$$x_{k,i} = F(y_{k,i}, t_{k,i}).$$

where

$$t_{k,i} = \frac{w_{k+1} + \ldots + w_n}{w_i + w_k + w_{k+1} + \ldots + w_n}.$$

Taking the derivative of both sides with respect to s, we get

$$\frac{d}{ds}x_{k,i} = \frac{d}{ds}F(y_{k,i}, t_{k,i}) \cdot \frac{d}{ds}y_{k,i}.$$

Setting $s = 1$, and using that $\frac{d}{ds}F(s,t)|_{s=1} = \frac{1-t}{2}$, we deduce that

$$a_{k,i} = \frac{(1 - t_{k,i})}{2} \cdot b_{k,i}, \tag{5.5}$$

for all $1 \leqslant i < k \leqslant n-1$.

Recall that

$$y_{k+1,i} = \frac{w_i x_{k,i} + w_k}{w_i + w_k x_{k,i}} \cdot \frac{y_{k,i}}{x_{k,i}}.$$

Taking the logarithmic derivative on both sides we get

$$\frac{\frac{d}{ds}y_{k+1,i}}{y_{k+1,i}} = \frac{w_i \frac{d}{ds}x_{k,i}}{w_i x_{k,i} + w_k} + \frac{\frac{d}{ds}y_{k,i}}{y_{k,i}} - \frac{w_k \frac{d}{ds}x_{k,i}}{w_i x_{k,i} + w_k} - \frac{\frac{d}{ds}x_{k,i}}{x_{k,i}}.$$

Setting $s = 1$, we get

$$b_{k+1,i} = \frac{w_i a_{k,i}}{w_i + w_k} + b_{k,i} - \frac{w_k a_{k,i}}{w_i + w_k} - a_{k,i}.$$

Let us use that $a_{k,i} = \frac{(1-t_{k,i})}{2} \cdot b_{k,i}$ to express the right hand side in terms of $b_{k,i}$ and the weights only. After a quick algebraic manipulation, we get

$$b_{k+1,i} = \frac{w_i + w_{k+1} + \ldots + w_n}{w_i + w_k + \ldots + w_n} \cdot b_{k,i}. \tag{5.6}$$

Finally, recall that

$$y_{k+1,k} = \frac{s}{\prod_{i=1}^{k-1} x_{k,i} \cdot \frac{w_i x_{k,i} + w_k}{w_i + w_k x_{k,i}}}.$$

Taking the logarithmic derivative, we get

$$\frac{\frac{d}{ds}y_{k+1,k}}{y_{k+1,k}} = \frac{1}{s} - \sum_{i=1}^{k-1} \Big(\frac{\frac{d}{ds}x_{k,i}}{x_{k,i}} + \frac{w_i \frac{d}{ds}x_{k,i}}{w_i x_{k,i} + w_k} - \frac{w_k \frac{d}{ds}x_{k,i}}{w_i x_{k,i} + w_k} \Big)$$

Now, setting $s = 1$, we get

$$b_{k+1,k} = 1 - \sum_{i=1}^{k-1} \Big(a_{k,i} + \frac{w_i a_{k,i}}{w_i + w_k} - \frac{w_k a_{k,i}}{w_i + w_k} \Big).$$

Let us use that $a_{k,i} = \frac{(1-t_{k,i})}{2} \cdot b_{k,i}$ on the right hand side. After a quick algebraic manipulation, we get

$$b_{k+1,k} = 1 - \sum_{i=1}^{k-1} \frac{w_i}{w_i + w_k + \ldots + w_n} b_{k,i}. \tag{5.7}$$

The equations (5.4), (5.6), and (5.7) define the $b_{k,i}$s recursively, while (5.5) expresses $a_{k,i}$ in terms of $b_{k,i}$. Using the recursive formulas for the $b_{k,i}$s, we can derive by a straightforward proof by induction that

$$b_{k,i} = \frac{w_i + w_k + w_{k+1} + \ldots + w_n}{w},$$

for all $1 \leqslant i < k \leqslant n$. Then, keeping in mind the definition of the $t_{k,i}$s, (5.5) yields that

$$a_{k,i} = \frac{w_i + w_k}{2w}.$$

Since the weights are positive, it is evident that $a_{k,i} > 0$ and $b_{k,i} > 0$ for all k, i, as desired. $\square$

Theorem 9. *Let $n \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that for all $0 \leq s \leq 1$ and all $m \in \mathbb{N}$ the total homothety $H(s)$ on $\mathcal{P}_{n,m}$ (with arbitrary weights associated to it) is a product of at most N pinches.*

Proof. Let $0 < \varepsilon < 1$ be as in Theorem 8. Choose $M \in \mathbb{N}$ large enough so that $\varepsilon^{\frac{1}{M}} > 1 - \varepsilon$. Now, if $\varepsilon \leqslant s \leqslant 1$ then $1 - \varepsilon < s^{\frac{1}{M}}$, and so $H(s^{\frac{1}{M}})$ is expressible as a product of $(n-1)^2$ pinches, by the previous theorem. Since $H(s) = (H(s^{\frac{1}{M}}))^M$, it follows that $H(s)$ is expressible as a product of $M(n-1)^2$ pinches. On the other hand, if $0 \leqslant s < \varepsilon$ then $H(s)$ is expressible as product of $(n-1)^2$ pinches, by the previous theorem. Therefore, $N = M(n-1)^2$ is as required. □

Corollary 10. *The set of all $Q \in \mathcal{P}_{n,m}$ attainable from a given $P \in \mathcal{P}_{n,m}$ by repeated applications of pinches is star shaped, with center the constant n-tuple $(G, \ldots, G)$, where G is the centroid of P.*

Proof. Since Q is attainable from P through pinches, it has the same centroid as P. Hence $H(0)Q = (G, \ldots, G)$. Thus, for all $0 \leqslant s \leqslant 1$,

$$H(s)Q = sQ + (1-s)(G, \ldots, G).$$

By Theorem 9, the left hand side is attainable from Q, whence also from P, through repeated applications of pinches. □

We do not know the answer to the following question, though, as mentioned above, we believe it to be affirmative:

Question 11. Given any $n \geqslant 4$ and any $0 \leqslant s \leqslant 1$, are the numbers $x_{k,i}$ and $y_{k,i}$ defined in (5.2) all in the interval $[0,1]$?

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