Expanding maps of the circle rerevisited: Positive Lyapunov exponents in a rich family

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Abstract. In this paper we revisit once again, see [**ShSu**], a family of expanding circle endomorphisms. We consider a family $\{B_\theta\}$ of Blaschke products acting on the unit circle, $\mathbb{T}$, in the complex plane obtained by composing a given Blashke product B with the rotations about zero given by mulitplication by $\theta \in \mathbb{T}$. While the initial map B may have a fixed sink on $\mathbb{T}$ there is always an open set of θ for which B_θ is an expanding map. We prove a lower bound for the average measure theoretic entropy of this family of maps in terms of $\int ln|B'(z)|dz$.

1. Introduction

Several papers have suggested the possibility of giving lower bounds for the average entropy or Lyapunov exponents in a rich enough family of dynamical systems [**BuPuShWi**], [**LSSW**]. A particular consequence would establish the existence of positive entropy for a positive measure set of parameters in terms of comparatively easily computable quantities. A linear algebra analogue is proven in [**DeSh**]. In this paper we accomplish the task for families of (finite) Blaschke products. In these families it is fairly easy to establish the existence of positive measure sets of parameters which define expanding maps of the circle. Here we give a lower bound for the average entropy of these expanding maps with respect to the natural invariant measures which are absolutely continuous with respect to Lebesgue measure.

A (finite) Blaschke product is a map of the form

$$B(z) = \theta_0 \prod_{i=1}^{n} \frac{z - a_i}{1 - z\overline{a_i}}$$

where $n \geq 2$, $a_i \in \mathbb{C}, |a_i| < 1$, $i = 1 \ldots n$ and $\theta_0 \in \mathbb{C}$ with $|\theta_0| = 1$. B is a rational mapping of $\mathbb{C}$, it is an analytic function in a neighborhood of the the unit disc $\mathbb{D}$, and B maps the unit circle $\mathbb{T}$ to itself. In this paper we consider the family of Blaschke products,

$$\{B_\theta\}_{\{\theta \in \mathbb{T}\}} = \{\theta B\}_{\{\theta \in \mathbb{T}\}}.$$

Theorem 1.1. *Given a family of Blaschke products $\{B_\theta\}_{\{\theta \in \mathbb{T}\}}$, one of the next two options holds for any $\theta \in \mathbb{T}$:*

(1) *B_θ is an expanding map, i.e.: there are $n = n(\theta)$, and $\lambda = \lambda(\theta) > 1$ such that*

$$|B_\theta^{n\prime}(x)| > \lambda;$$

(2) *B_θ has a unique attracting or indifferent fixed point in $\mathbb{T}$.*

Research partially supported by an NSERC Discovery Grant.

Moreover, the set of $\theta \in \mathbb{T}$ satisfying the first option is a nonempty open set.

In the next theorem, we relate the previous options to the statistical behaviour of B_θ. Let λ be Lebesgue measure on $\mathbb{T}$ normalized to be a probability measure, $\lambda(\mathbb{T}) = 1$.

THEOREM 1.2. *Given a family of Blaschke products $\{B_\theta\}_{\{\theta\in\mathbb{T}\}}$ follows that for all θ, the push forwards of Lebesgue measure $B^n_{\theta\star}(\lambda)$, converge to a measure μ_θ which is:*

(1) *absolutely continuous with respect to Lebesgue if B_θ satisfies condition 1 of theorem 1.1, or*
(2) *a Dirac delta measure supported on an attracting or indifferent fixed point of B_θ on $\mathbb{T}$.*

As a consequences of theorem 1.2 it follows that for any θ we can define the metric entropy, h_θ, of B_θ with respect to μ_θ and it satisfies

$$h_\theta = \int_{\mathbb{T}} ln|B'(z)|d\mu_\theta$$

when it is positive.

In the next theorem we give a lower bound for the average measure theoretic entropy of this family of maps in terms of $\int ln|B'(z)|dz$.

THEOREM 1.3. *Given a family of Blaschke products $\{B_\theta\}_{\{\theta\in\mathbb{T}\}}$ it follows that:*

A)
$$\int h_\theta d\theta \geq \int_{\mathbb{T}} ln|B'(z)|dz$$
with equality if and only if $|B'(z)| \geq 1$ for all $z \in \mathbb{T}$.

B) *More precisely,*
$$\int h_\theta d\theta = \int_{\mathbb{T}} ln^+|B'(z)|dz + \int_{\mathbb{T}} |B'(z)|ln^-|B'(z)|dz.$$

Here ln^+ equals ln when it is positive and zero otherwise while ln^- equals ln when it is negative and zero otherwise. When h_θ is positive it equals the Lyapunov exponent of B_θ with respect to μ_θ; i.e.: for almost every point with respect to Lebesgue measure

$$h_\theta = \lim_{n\to+\infty} \frac{1}{n} ln|B^{n\prime}(z)|.$$

So we could equally well state our results with respect to Lyapunov exponents.

Most of the proof of the previous theorems could be assembled from results already in the literature. We give an alternate largely self contained proof in the next sections. The proof consists of three parts:

1) For all θ, theorem 1.1 or 1.2 holds.
2) $\int B^n_{\theta\star}(\lambda)d\theta = \lambda$ for all n.
3) Let $\phi : \mathbb{T} \to \mathbb{R}$ be continuous.
   Then $\int_{\mathbb{T}} \phi d\lambda = \int(\int_{\mathbb{T}} \phi d\mu_\theta(z))d\theta$.

The proof is completed by applying 3) to $ln|B'(z)|$ applying 1) and 2) and changing variables for those θ for which μ_θ is supported on a contracting or indifferent fixed point. This proves B) and A) follows. The proof is carried out in detail in the next sections.

2. The fixed points of B.

For any Blaschke product B as above the equation $z = B(z)$ has at most $n + 1$ zeros in the complex plane, $\mathbb{C}$. So $B : \mathbb{C} \to \mathbb{C}$ has at most $n + 1$ fixed points in $\mathbb{C}$. The map $B : \mathbb{T} \to \mathbb{T}$ has degree n. By the Lefschetz formula B has -(n-1) fixed points counted with index on $\mathbb{T}$. Thus B has at least (n-1) expanding fixed points on $\mathbb{T}$ and at most (n+1) fixed points in all.

PROPOSITION 2.1. *One of the following three mutually exclusive cases holds:*

(a) B has all its fixed points on $\mathbb{T}$. There is exactly one of them z_0 that is a sink and the other n are expanding.

(b) B has $n-1$ fixed points on $\mathbb{T}$, all expanding. It has one fixed point inside the disc which is a sink, and one outside. These two fixed points are related by the formula $z_0 \to \frac{1}{\overline{z_0}}$ (hence, they lie on the same ray passing through the origin).

(c) B has all its fixed points on $\mathbb{T}$. There is one that is an indifferent saddle node fixed point, $B(z_0) = z_0$ and $B'(z_0) = 1$.

In all three cases there is an open set of points in the disc which tend to z_0 under iteration of B.

PROOF. If $B'(z) \neq 1$ for all fixed points of B on the circle, by the Lefschetz formula we can have n expanding fixed points and one sink on the circle or $n-1$ expanding points on the circle. In the first case we are in situation (a). In the second case, since $B(z)\overline{B(\overline{z}^{-1})} = 1$ for all $z \in \mathbb{C}$, we must be in case (b). The fact that the fixed point in the interior of the disc is attracting follows from direct calculation or the Schwarz lemma. Case (c) represents the remaining cases.

□

Iterates of B. The sequence $B^{(n)}(z)$ is uniformly bounded in the unit disc (i.e., it is a normal family). Let z_0 be the attracting or indifferent fixed point in Proposition 2.1. Observe that a sink or an indifferent fixed point of a rational mapping of $\mathbb{C}$ always attracts an open set of points. Therefore there is an open set of points in which $\{B^{(n)}(z)\}_n$ converges uniformly to z_0. Thus, by Vitali's convergence Theorem the sequence $\{B^{(n)}(z)\}_n$ converges uniformly on compact sets of the open unit disc to z_0. Thus $B^n(z) \to z_0$ for any z in the open unit disc.

Incidentally, this proves that the fixed point z_0 described in Proposition 2.1 is unique in the closed unit disc as an attracting or indifferent fixed point.

***B* composed with rotations.** We now consider the one parameter family of functions $B_\theta = \theta B$. Our main interest will be when θ goes around the circle, but we will also consider c taking values in the disc, $\mathbb{D}$.

For every θ consider the set of fixed points of B_θ. As θ goes around the circle the fixed points of B_θ will be in situations (a), (b) or (c) described before. Case (c) will happen at most a finite number of times. For every $\theta \in \mathbb{T}$ we define $\alpha(\theta)$ as the unique sink of B if we are in situations (a) or (b). In case (c) $\alpha(\theta)$ is the unique indifferent fixed point of B_θ(but in fact this case is irrelevant for our ultimate discussion because it is measure zero in the parameter). For all $z_0 \in \mathbb{T}$ such that $|B'(z_0)| \leq 1$ there is one value of θ (namely $\theta = z_0/B(z_0)$), such that z_0 is a fixed sink or indifferent point of B_θ. Thus, all these values belong to the range of α. Finally, if $|c| < 1$ we define $\alpha(c)$ as the unique fixed point of B_c inside the unit disc.

PROPOSITION 2.2. *The function α is analytic in the open unit disc and continuous in the closed unit disc.*

PROOF. By the implicit function theorem the attracting fixed points of B_θ vary analytically with θ in the closed disc minus the finite set of θ for which B_θ has an indifferent fixed point in $\mathbb{T}$, the values of which provide a continuous extension of the function.

□

The next corollary is an obvious extension of our discussion of iterates to B_c for $|c| \in \mathbb{D}$

COROLLARY 2.3. *Let z_0 be inside the open unit disc and c in the closed disc. Then $B_c^{(n)}(z_0)$ converges to $\alpha(c)$.*

3. Expanding maps and proof of theorem 1.1.

PROPOSITION 3.1. *If $\theta_0 \in \mathbb{T}$ and $\alpha(\theta_0)$ is in the open unit disc, then there is an $n > 0$ such that $|B^{n\prime}_{\theta_0}(z)| > 1$ for all $z \in \mathbb{T}$. That is, B_{θ_0} is expanding.*

PROOF. Suppose z_0 is a fixed point of B_{θ_0} inside the disc. Let C_r be a disk of radius r, $r < 1$ and center 0 that contains z_0. Since $B^n_{\theta_0}$ converges uniformly to z_0 there is some n such that $B^{(n)}_{\theta_0}(C_r) \subset C_r$. This implies that $\theta B^n_{\theta_0}$ has a fixed point in C_r for all $\theta \in \mathbb{T}$. This means that $B^n_{\theta_0}$ never has an attracting or indifferent fixed point on the unit circle ; hence, the set $\{z \in \mathbb{T} : |B^{n\prime}_{\theta_0}(z)| \leq 1\}$ is empty. □

Observe that this finishes the first part of theorem 1.1. In fact, if the attracting fixed point of B_θ is in the open unit disc then the map is expanding; if not, it has a unique fixed point in the circle which is either attracting or an indifferent saddle-node point. Now we proceed to finish the proof of theorem 1.1.

PROOF. By proposition 3.1 it is enough to show that that there exists θ_0 such that $\alpha(\theta_0)$ is in the open unit disc. Let us assume that there is x_0 such that $|B'(x_0)| = 1$ (otherwise, the thesis of the theorem holds for every $\theta \in \mathbb{T}$). Therefore, there exists θ_0 such that $B_{\theta_0}(x_0) = x_0$ and so x_0 is an indifferent saddle-node. This implies that there is $\epsilon_0 > 0$ and an open interval J_0 in $\mathbb{T}$ containing x_0 such that either for every $\theta \in (\theta_0, \theta_0 + \epsilon_0)$ B_θ does not have a fixed point in J_0 and for every $\theta \in (\theta_0 - \epsilon_0, \theta_0)$ B_θ has a sink in J_0, or for every $\theta \in (\theta_0 - \epsilon_0, \theta_0)$, B_θ does not have a fixed point in J_0 and for every $\theta \in (\theta_0, \theta_0 + \epsilon_0)$ B_θ has a sink in J_0. Let us assume that the first option hold. To conclude the theorem, it is enough to show that there exists ϵ_1 such that for every $\theta \in (\theta_0, \theta_0 + \epsilon_1)$ B_θ does not have a sink or an indifferent fixed point in the complement of J_0. If not, there is a sequence $\theta_n \to \theta_0$ such that B_{θ_n} has a sink or indifferent fixed point contained in J_0^c. But then so does B_{θ_0} which contradicts the uniqueness of the fixed point x_0 among indifferent or attracting fixed points of B_{θ_0}

□

4. Push forwards of Lebesgue measure. Proof of theorem 1.2 and 1.3.

If B has a fixed point z_0 on the circle then the Dirac measure, μ_{z_0}, corresponding to that point is left invariant by B. Given a point z_0 in the interior of the unit disc we let μ_{z_0} denote the absolutely continuous measure on the circle $\mathbb{T}$ defined in any of three equivalent ways:

- Let $h : \mathbb{T} \to \mathbb{C}$ be continuous and $\tilde{h}$ its harmonic extension to the disc. Then $\int_{\mathbb{T}} h d\mu_{z_0} = \tilde{h}(z_0)$.
- Let $\int_{\mathbb{T}} h d\mu_{z_0} = \int_{\mathbb{T}} h P z_0 d\lambda$ where P_{z_0} is the Poisson kernel and λ is Lebesgue measure.
- Let A_{z_0} be a fractional linear transformation mapping 0 to z_0. Then $\mu_{z_0} = A_{z_0\star}(\lambda)$.

PROPOSITION 4.1. *Let B be a Blaschke product. Then $B_\star(\mu_{z_0}) = \mu_{B(z_0)}$. Thus if B has a fixed point z_0 inside the disc then the absolutely continuous measure given by μ_{z_0} is left invariant by B.*

PROOF. Let $h : \mathbb{T} \to \mathbb{C}$ be continuous and $\tilde{h}$ its harmonic extension to the disc. Then $\int_{\mathbb{T}} h dB_\star(\mu_{z_0}) = \int_{\mathbb{T}} h \circ B d(\mu_{z_0}) d\lambda = \widetilde{h \circ B}(z_0)$. Since B is analytic $\tilde{h} \circ B$ is harmonic, thus $\widetilde{h \circ B}(z_0) = \tilde{h} \circ B(z_0) = \int_{\mathbb{T}(\mathbb{C})} h d\mu_{B(z_0)}$. □

For every $c \in \mathbb{D}$ we write $\nu_c = \mu_{\alpha(c)}$. Then $|\alpha(c)| < 1$ if and only if ν_c is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$ and for $\theta \in \mathbb{T}$ follows that $|\alpha(\theta)| < 1$ if and only if B_θ is expanding. If $|\alpha(\theta)| = 1$ then ν_θ is the Dirac measure supported on $\alpha(\theta)$.

We are now ready to prove theorem 1.2:

PROOF. By corollary 2.3 $B^{(n)}_\theta(0)$ converges to $\alpha(\theta)$. It follows that $B^n_{\theta\star}(\lambda)$ converges to the measure ν_θ defined above. When B_θ is expanding then, ν_θ is absolutely continuous with respect to Lebesgue, and

$h_\theta = \int_{\mathbb{T}} ln|B'_\theta(z)|d\nu_\theta = \int_{\mathbb{T}} ln|B'(z)|d\nu_\theta$ (see [**L**]). In the case that B_θ has an attracting or indifferent fixed point, follows that the push forward converge to a Dirac measure supported on this point.

□

REMARK 4.2. Theorem 1.2 has a version for C^2 dynamical systems which we could have used here with a little work (see [**M**]).

Now we prove item 2) of the introduction.

PROPOSITION 4.3. $\int_{\mathbb{T}} B^n_{\theta\star}(\lambda)d\theta = \lambda$ *for all* n.

PROOF. For any continuous function $h : \mathbb{T} \to \mathbb{R}$, $\int \int_{\mathbb{T}} hdB^n_{\theta\star}(\lambda)d\theta = \int \tilde{h}(B^n_\theta(0))d\theta$ and since the map $c \to B^n_c(0)$ is an analytic function of c in the unit disc and at $c = 0$, $B^n_c(0) = 0$ follows that $\int \tilde{h}(B^n_\theta(0))d\theta = \tilde{h} \circ B^n(0) = \tilde{h}(0)$.

□

REMARK 4.4. Propositon 4.3 can also be proved also proven by Fourier series as was done in 4.11 and 4.12 of [**LSSW**].

Finally we finish the proof theorem 1.3.

PROPOSITION 4.5. *Let* $\phi : \mathbb{T} \to \mathbb{R}$ *be continuous. Then* $\int_{\mathbb{T}} \phi d\lambda = \int (\int_{\mathbb{T}} \phi d\nu_\theta(z))d\theta$.

PROOF. By the Lebesgue dominated convergence theorem $\int (\int_{\mathbb{T}} \phi d\nu_{\theta(z)})d\theta = lim \int (\int_{\mathbb{T}} \phi dB^n_{\theta\star}(\lambda))d\theta = \int_{\mathbb{T}} \phi d\lambda$ □

Now we proceed to give the proof of theorem 1.3.

PROOF. We consider the set $\mathbb{T}_l = \{\theta \in \mathbb{T}|\nu_\theta \text{is absolutely continuous}\}$ and $\mathbb{T}_d = \{\theta \in \mathbb{T}|\nu_\theta \text{is Dirac}\}$. $\mathbb{T}_e = \{\theta \in \mathbb{T}|B_\theta \text{is expanding}\}$ and $\mathbb{T}_a = \{z \in \mathbb{T}||B'(z)| \leq 1\}$.

$$\int_{\mathbb{T}} ln|B'(z)|d\lambda =$$
$$\int_{\mathbb{T}_l} ln|B'(z)|d\nu_\theta d\theta + \int_{\mathbb{T}_d} ln|B'(\alpha(\theta))|d\theta =$$
$$\int_{\mathbb{T}_e} h_\theta d\theta + \int_{\mathbb{T}_a} (1 - |B'(z)|)ln|B'(z)|d\lambda$$

where this last equality follows from the fact that $d\theta = (1 - |B'(z)|)d\lambda$. Finally, subtract the last term on the right from the term on the left to prove the theorem.

□

5. Remarks, Questions and Conclusions

We have given lower bound and exact integral estimates for the average entropy of a family of Blaschke products with respect to the SRB measures determined by iterates of members of the family. In [**LSSW**] similar estimates for a family of diffeomorphishms of the sphere were discussed, but nothing positive was proven for deterministic products as were considered here. The success with Blaschke products suggests other families of examples.

1) What about the family θf where f is an immersion of the circle of finite smoothness, or even a C^r topological covering with a cubic singularity?

2) What about similar estimates for the quadratic family of maps of the unit interval, normalized to have the unit interval as the image? Is there a meaningful measure on the space of parameters which is absolutely continuous with respect to Lebesgue and for which positive estimates of the mean entropy can be relatively easily proven?
3) Let A_1, A_2 and A_3 be fractional linear tranformations of the unit disc. Let $(\theta, \psi) \in \mathbb{T} \times \mathbb{T}$ and consider the family of diffeomorphism of the two torus $\mathbb{T} \times \mathbb{T}$ defined by
$$B_{\theta,\psi}(w, z) = (\theta A_1(w)A_2(w)\psi A_3(z), \theta A_2(w)\psi A_3(z)).$$
Then these diffeomorphisma are all isotopic to the usual linear Anosov diffeomorphism of the two torus, which is usually written additively (in the Anosov case, $A_1(w) = A_2(w) = w$ and $A_3(z) = z$). Can one estimate the average entropy of SRB measures associated to this family of diffeomorphism? Is the set of (θ, ψ) for which $B_{\theta,\psi}$ is Anosov non-empty? Is the set of (θ, ψ) for which $B_{\theta,\psi}$ has an SRB measure of positive entropy of positive measure?
4) Our theorem involves a probability measure μ on a space of parameters P of dynamical systems of a manifold M with a probability measure ν. What can be said about the existence of measures satisfying: For almost all $p \in P$, $lim \frac{1}{n} \sum f^j_{p\star}(\nu)$ converges to a measure ν_p and $lim \frac{1}{n} \sum \int f^j_{p\star}(\nu)d\mu = \nu$? or even as in item 2) of the introduction that $\int f^n_{p\star}(\nu)d\mu = \nu$ for all n?

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