Proof. Let $z = (z_1, z_2, \dots, z_m)$ be a vector of formal variables. For $v \in \mathbb{N}^m$ (the natural numbers start in zero) we write $z^v = z^{v_1} z^{v_2} \dots z^{v_m}$. For a subset $S \subset \mathbb{N}^m$ we write $S(z) = \sum_{i \in S} z^i$ and call $S(z)$ the formal series of the set S. The proof rests on the following marvelous theorem.

Theorem 1. *Let Φ be an $m \times m$ matrix with integer coefficients. Let us denote by S the set of solutions of the system $\Phi x = 0$ such that $x \in \mathbb{N}^m$. The formal series $S(z)$ is a rational function. Its denominator has the form $\Pi_i (1 - z^{\alpha_i})$.*

By some elementary analysis, we can extend the previous theorem to sets not necessarily described by a system of homogeneous diophantine equations. We can add to the system $\Phi x = 0$ inequalities of the form $ax \leq 0$. The resulting system is transformed into a homogeneous system of equalities introducing slack variables. The generating function of the original set is obtained setting to 1 the formal variables corresponding to the slack variables. Thus, the formal series of the new set will still be rational. We can also add inequations to the system since $\{x \in S : ax \neq 0\} = S \backslash \{x \in S : ax = 0\}$. Finally, by the principle of inclusions and exclusions, we can take unions of sets defined by homogeneous systems of equations, inequalities and inequations. Thus, if the set S is described by a combination of equalities, inequalities and inequations concatenated by $\vee$s (ORs) and $\wedge$s (ANDs) then $S(z)$ is rational.

Let us identify the $n \times n$ chessboard with the set $[0, n-1]^2$. Let $F = \{(\alpha_i, \beta_i), i = 1, \dots, kw\}$ be a figure of type (n, k, w). The fact that there are exactly w squares on the same row as (α_1, β_1) can be expressed by the system

$$\bigvee_{P_{1,w}} (\alpha_1 = \alpha_s \wedge \alpha_1 \neq \alpha_{s'}, \forall s \in P_{1,w}, \forall s' \notin P_{1,w}),$$

where $P_{1,w}$ runs through all subsets of $\{2, \dots, kw\}$ with exactly $w-1$ elements. We add similar equations for the columns and diagonals passing through (α_1, β_1) and similarly for every other square of the figure. To account for the fact that the figure is contained in a finite chessboard of size n we add an extra variable γ and the inequalities

$$\alpha_i \leq \gamma, \; \beta_i \leq \gamma, \quad i = 1, \dots, kw.$$

Let S be the set of vectors $(\alpha_1, \beta_1, \dots, \alpha_{kw}, \beta_{kw}, \gamma) \in \mathbb{N}^{2kw+1}$ satisfying the preceding system. By Theorem 1 and the remark below it the formal series $S(z)$ is a rational function. Setting the formal variables corresponding to $\{(\alpha_i, \beta_i), i = 1, \dots, kw\}$ equal to 1 we get that

$$\sum_{n=0}^{\infty} W(n, k, w) z_\gamma^{n-1}$$

is also a rational function. □