Figures of Constant Width on a Chessboard

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1. INTRODUCTION. Consider the following figure in a 5×5 chessboard:

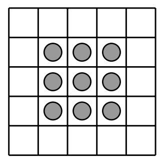

Figure 1. Every row and column contains three occupied squares.

(By "figure" we mean the subset of all the occupied squares of the chessboard.) It has the property that every row and column of the board that intersects the figure contains exactly three occupied squares.

Now look at the following figure in a 4×4 chessboard:

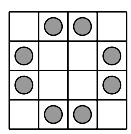

Figure 2. Every row, column and diagonal contains two occupied squares.

Every row and column of the board has only two occupied squares. But if we also consider the diagonals with slope $+1$ or -1, we see that each of them contains either zero or two squares of the figure.

We say that Figure 1 has constant width three by rows and columns, and that Figure 2 has constant width two by rows, columns, *and diagonals*.

The first figure is easy to generalize in the sense that the figure formed by all the squares of a $w \times w$ chessboard has constant width w by rows and columns. On the other hand, if we also consider the two diagonal directions as in Figure 2, the problem of finding figures of constant width higher than two becomes considerably harder. Is it possible, for example, to find a nonempty figure on some $n \times n$ chessboard such that every row, column, and diagonal intersects it in zero or three squares? This is a fun problem to think about. We encourage the reader to try it before he or she continues reading. As we will show here, the answer turns out to be yes, even for widths higher than three. The impatient reader might want to take a look at Figures 6, 11, 12, and 13.

In order to state our main result in a precise form, we introduce some terminology. We designate as a *figure* any set of squares in an $n \times n$ chessboard. A figure F has *constant width* w if every row, column, or diagonal intersects it in 0 or w squares. To be more exact, F is of *type* (n, k, w) if it has constant width w in a chessboard of size $n \times n$ and has kw squares. Observe that k is also the number of nonempty rows (or columns) in the chessboard.

Notice that the constant width figures of type $(n, n, 1)$ are the solutions of the n-queens problem. These are the configurations of n queens in an $n \times n$ chessboard such that none of them can attack any other. It is known that these configurations exist when $n \geq 4$ (see [**1**] and [**4**]).

In this article we prove the following theorem:

Theorem 1. *For every w there are constant width figures of type (n, k, w) for all pairs (n, k) with $n \geq k$ and k sufficiently large.*

2. CONSTRUCTING NEW FIGURES FROM OLD ONES. We identify the squares of an $n \times n$ chessboard with the elements of the set $\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-1\}$, and figures with the subsets of this set.

In some of the following constructions we make use of *figures of extended constant width w*. These are figures such that every row, column, diagonal, or extended diagonal intersects it in 0 or w squares. An *extended diagonal* is a set of squares with coordinates (i, j) such that either $i + j \equiv d \pmod{n}$ or $i - j \equiv d \pmod{n}$ for some d in $\{0, 1, \ldots, n-1\}$. For example, Figure 2 is not of extended constant width, but Figure 3 is. One of the extended diagonals is indicated as a dashed line.

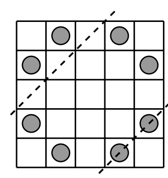

Figure 3. Extended constant width figure of type (5, 4, 2) and extended diagonal.

Notice that the extended constant width figures of type $(n, n, 1)$ are the toroidal solutions of the n-queens problem. It is known that these figures exist when $\gcd(n, 6) = 1$ (see [**1**, pp. 363–374] and [**4**]).

Composing figures. Let F_1 be a figure in an $n \times n$ chessboard, and F_2 a figure in an $m \times m$ chessboard. We construct the *composition* $F_1 \circ F_2$ of F_1 and F_2 by dividing an $nm \times nm$ chessboard into squares of size $n \times n$ and placing a copy of F_1 in each of the squares belonging to F_2 (see Figure 4). This construction has been used in papers about the n-queens problem (see [**1**] and [**4**]).

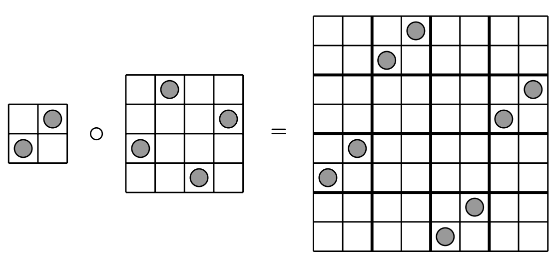

Figure 4. Example of composition of two figures.

The composition of two constant width figures is not necessarily of constant width. Nevertheless, we have the following lemma.

Lemma 1. *Let F_1 and F_2 be constant width figures of types (m, k, w_1) and (n, l, w_2), respectively. Then the following statements are true:*

(a) *If F_1 is of extended constant width, then $F_1 \circ F_2$ is a constant width figure of type $(mn, kl, w_1 w_2)$.*

(b) *If both F_1 and F_2 are of extended constant width, then $F_1 \circ F_2$ is of extended constant width.*

Proof. The composition of two figures admits the following arithmetic description. Let (x, y) belong to $\{0, \dots, mn-1\} \times \{0, \dots, mn-1\}$ and write $x = i_1 + m i_2$ and $y = j_1 + m j_2$, with i_1 and j_1 in $\{0, \dots, m-1\}$ and i_2 and j_2 in $\{0, \dots, n-1\}$. Then

$$(x, y) \in F_1 \circ F_2 \quad \Leftrightarrow \quad (i_1, j_1) \in F_1,\ (i_2, j_2) \in F_2.$$

It is clear that $F = F_1 \circ F_2$ occupies $(k w_1)(l w_2)$ squares on a chessboard of size $mn \times mn$. We need to prove that F has constant width $w_1 w_2$.

Consider the width of F by columns. Let c belong to $\{0, \dots, mn-1\}$ and write $c = c_1 + m c_2$. We have

$$x = c, \quad (x, y) \in F \quad \Leftrightarrow \quad \begin{cases} i_1 = c_1, \quad (i_1, j_1) \in F_1; & (1) \\ i_2 = c_2, \quad (i_2, j_2) \in F_2. & (2) \end{cases}$$

Equations (1) and (2) have 0 or w_1 and 0 or w_2 solutions, respectively. Thus the number of solutions of $x = c$ with x in F is 0 or $w_1 w_2$; that is, F has constant width $w_1 w_2$ by columns. The same argument applies to rows.

For the diagonal directions we need to consider parts (a) and (b) separately. For (a) look at the diagonals of slope $+1$. For d in $\{-mn+1, \dots, mn-1\}$ we have

$$x - y = d, \quad (x, y) \in F \quad \Leftrightarrow \quad \begin{cases} i_1 - j_1 \equiv d \pmod{m}, \quad (i_1, j_1) \in F_1; & (3) \\ i_2 - j_2 = \dfrac{d - (i_1 - j_1)}{m}, \quad (i_2, j_2) \in F_2. & (4) \end{cases}$$

Since F_1 is an extended constant width figure, equation (3) has 0 or w_1 solutions. For each of these solutions equation (4) has 0 or w_2 solutions. Thus F has constant width $w_1 w_2$ by diagonals of slope $+1$. The proof for the diagonals of slope -1 is analogous.

Turning to (b), consider the extended diagonals of slope $+1$. For d in $\{0, \dots, mn-1\}$ we have

$$\begin{aligned} &x - y \equiv d \pmod{mn}, \\ &(x, y) \in F \Leftrightarrow \begin{cases} i_1 - j_1 \equiv d \pmod{m}, & (i_1, j_1) \in F_1; \\ i_2 - j_2 \equiv \dfrac{d - (i_1 - j_1)}{m} \pmod{n}, & (i_2, j_2) \in F_2, \end{cases} \end{aligned}$$

and the reasoning follows as before. The proof for the extended diagonals of slope -1 is analogous. ■

As a consequence of Lemma 1, we obtain a simple way of constructing a figure of constant width four. Composing the extended constant width figure of type $(5, 4, 2)$ shown in Figure 3 with the constant width figure of type $(4, 4, 2)$ shown in Figure 2, we obtain a constant width figure of type $(20, 16, 4)$.

Transversals. A figure T is a *transversal* of a figure F if T is a subset of F and every row, column, or diagonal that intersects F intersects T in exactly one square. For example, Figure 5 shows two transversals of Figure 2. If a constant width figure F of type (n, k, w) can be decomposed into transversals (i.e., F is the disjoint union of w transversals), then we can delete $w - w'$ of those transversals to obtain a constant width figure of type (n, k, w').

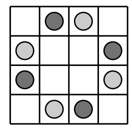

Figure 5. Two disjoint transversals of the constant width figure of type (4, 4, 2).

It is a corollary of Lemma 1 that if T_1 and T_2 are transversals of F_1 and F_2, respectively, then $T_1 \circ T_2$ is a transversal of $F_1 \circ F_2$. Hence, if F_1 and F_2 are decomposable into transversals, so is $F_1 \circ F_2$.

Now we can construct a figure of constant width three. Since both Figures 2 and 3 are decomposable into two transversals, the constant width figure of type (20, 16, 4) that we constructed earlier can be decomposed into four transversals. Deleting one of them, we obtain the constant width figure of type (20, 16, 3) shown in Figure 6.

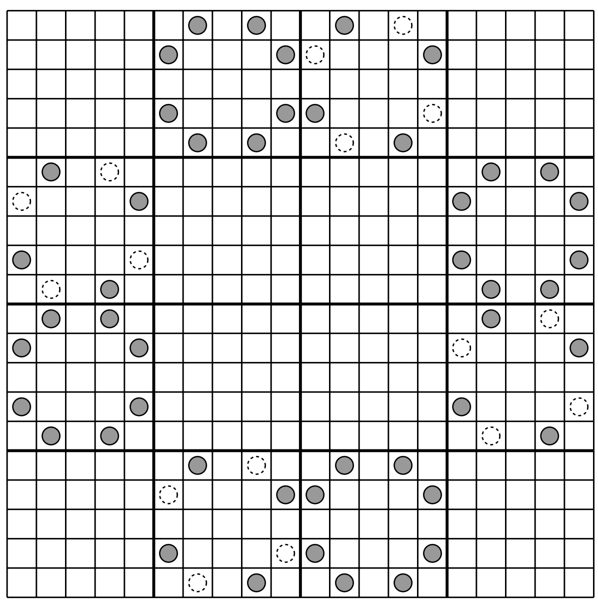

Figure 6. Constant width figure of type (20, 16, 3) obtained by deleting a transversal from a constant width figure of type (20, 16, 4).

By the way, not every figure of constant width w can be decomposed into w transversals (see, for example, Figure 11). This is because we are considering diagonals in the definition of constant width figure. If we consider only rows and columns, König's theorem [**3**, p. 188] guarantees that there is always such a decomposition.

Adding figures. Let F_1 be an extended constant width figure of type (n_1, n_1, w) and F_2 a constant width figure of type (n_2, n_2, w), where $n_1 > n_2$. We construct the *addition* F of F_1 and F_2 by placing four copies of F_1 and one copy of F_2 on a $(4n_1 + n_2) \times (4n_1 + n_2)$ chessboard, as shown in Figure 7. Under some conditions, this construction gives a constant width figure of type $(4n_1 + n_2, 4n_1 + n_2, w)$. It is clear that F has width w by rows and columns. Since F_1 is an extended constant width figure, F also has width w in the diagonals of slope $+1$ and in the diagonals of slope -1 outside the shaded area. Hence, F will have constant width w if the shaded area does not contain squares from the copies of F_1.

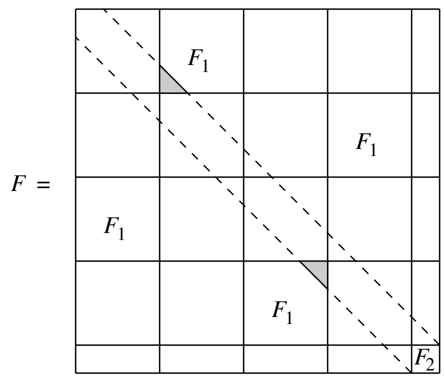

Figure 7. F is the addition of F_1 and F_2.

3. CONSTRUCTION OF CONSTANT WIDTH FIGURES OF TYPE (n, k, w).

We now take up the proof of Theorem 1. If $k < n$, every constant width figure of type (k, k, w) can be embedded into an $n \times n$ chessboard, say, in the upper left corner. In this way we get a constant width figure of type (n, k, w). Thus, it is enough to prove that there are constant width figures of type (n, n, w) for n sufficiently large.

The basic idea for finding figures of constant width w is the one that we used to construct the figure of constant width three. But that construction had the shortcoming that $k < n$. To overcome this difficulty, we need extended constant width figures with $n = k$ as our building blocks.

Lemma 2. *There exist extended constant width figures of types* $(13, 13, 2)$ *and* $(17, 17, 2)$ *that are decomposable into two transversals.*

Proof. The reader can verify that the figures shown in Figure 8 satisfy all the stated requirements. In each case, the two transversals are indicated in different gray tones. ■

Let A_1 be the figure of type $(13, 13, 2)$ shown in Figure 8, and define $A_n = A_{n-1} \circ A_1$ for $n \geq 2$. Then A_n is an extended constant width figure of type $(13^n, 13^n, 2^n)$ and can be decomposed into 2^n transversals. Let n_0 be the least number such that $2^{n_0} \geq w$. Deleting $2^{n_0} - w$ transversals from A_{n_0}, we get an extended constant width figure A of type (a, a, w), where $a = 13^{n_0}$. In the same way, using the figure of type $(17, 17, 2)$ shown in Figure 8, we obtain an extended constant width figure B of type (b, b, w), with $b = 17^{n_0}$.

Let T_x be an extended constant width figure of type $(x, x, 1)$ and F_y a constant width figure of type $(y, y, 1)$. Under certain conditions we can add $F_1 = T_x \circ A$ and $F_2 = B \circ F_y$ to obtain a constant width figure of type $(4xa + yb, 4xa + yb, w)$. It is

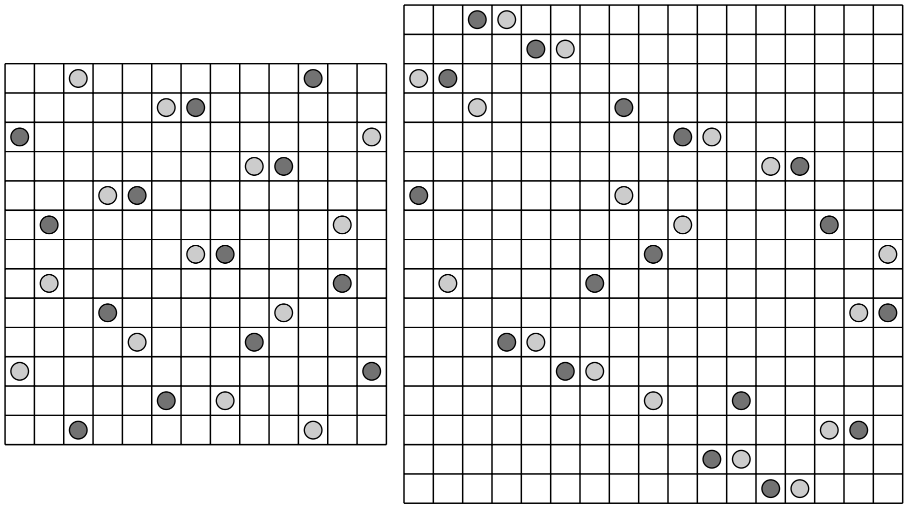

Figure 8. Extended constant width figures of types (13, 13, 2) and (17, 17, 2).

then an exercise in number theory to prove that any sufficiently large n can be written in the form $4xa + yb$, and with this the proof will be complete. In what follows we formalize these ideas.

As we mentioned before, T_x and F_y exist when $\gcd(x, 6) = 1$ and $y \geq 4$. Since T_x is an extended constant width figure, Lemma 1 ensures that F_1 is an extended constant width figure of type (ax, ax, w). By the same lemma, F_2 is a constant width figure of type (by, by, w) (but not necessarily an extended constant width figure).

If we also have $x > yb$, then the necessary conditions to add F_1 and F_2 are met. To see this, notice first that the four corners of A are empty squares, so the four $x \times x$ blocks in the corners of $F_1 = T_x \circ A$ are empty. Thus, the shaded area in Figure 7 does not contain squares of the copies of F_1. Figure 9 illustrates the argument.

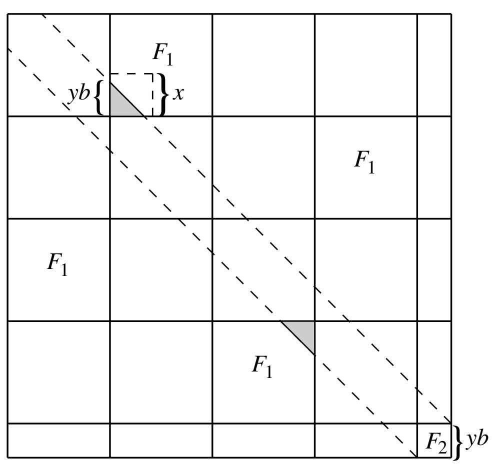

Figure 9. $x > yb$.

Notice that the a and b defined earlier satisfy $\gcd(24a, b) = 1$. Hence the following lemma completes the proof of Theorem 1.

Lemma 3. *If* $\gcd(24a, b) = 1$*, then there exists* N *such that any* n *greater than* N *can be written in the form* $n = 4xa + yb$ *with* $\gcd(x, 6) = 1$*,* $y \geq 4$*, and* $x > yb$*.*

Proof. Since $\gcd(24a, b) = 1$, for every n' satisfying $n' > 24ab - 24a - b$ we can find nonnegative integers x' and y' such that $n' = 24ax' + by'$. Moreover, replacing y' with its remainder modulo $24a$, we can assume that $y' < 24a$.

We now take $n = n' + 4(a + b)$, $x = 6x' + 1$, and $y = y' + 4$. We then have $n = 4ax + by$, $\gcd(x, 6) = 1$, and $4 \leq y \leq 24a + 4$. Such x and y exist whenever $n > 24ab - 20a + 3b$.

Finally, let $N = 4ab(24a + 4) + b(24a + 4)$ and consider any n greater than N. Since $n > 24ab - 20a + 3b$, we can choose x and y as before. Then

$$n = 4ax + by > 4ab(24a + 4) + b(24a + 4)$$

together with $y \leq 24a + 4$ implies that $x > b(24a + 4) > by$. We have thereby exhibited x and y with the properties required in the lemma. ∎

We conclude this section with some remarks on the size of the figures of constant width that we have constructed. The figure A is an extended constant width figure of type (a, a, w) with the property that it can be decomposed into w transversals. An estimate for a is $a = 13^{n_0} = O(w^{\log_2 13}) = O(w^{3.70\ldots})$. Also, we found in Lemma 3 a number N such that there are constant width figures of type (n, k, w) whenever $n \geq k \geq N$. This number satisfies $N = O(a^2 b)$, hence, $N = O(w^{\log_2 13^2 17}) = O(w^{11.48\ldots})$.

4. COMPUTATIONAL RESULTS.

In section 2 we were able to construct a figure of constant width three in a 20×20 chessboard. Also, the simple composition shown in Figure 10 gives a constant width figure of type $(81, 27, 3)$. But what if we want smaller figures?

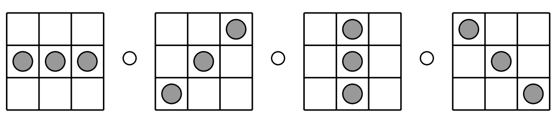

Figure 10. Constant width figure of type (81, 27, 3).

The authors have created a computer program that allows one to find smaller solutions, such as the one shown in Figure 11.

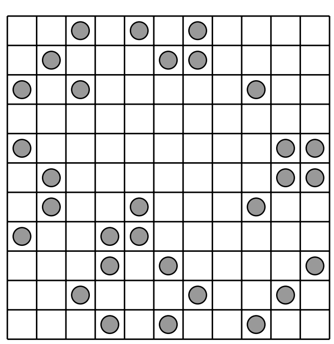

Figure 11. Constant width figure of type (11, 10, 3).

Finding figures of constant width with the computer is an interesting problem in its own right. The n-queens problem is usually solved using some refinement of the backtrack algorithm. Our case seems to be more complex, and finding all the solutions for a given (n, k, w) is too ambitious. Backtracking does not give good results even if one wants only a single solution.

To address the problem, we used a simulated annealing optimization. This is a class of stochastic algorithms commonly applied to solve combinatorial optimization problems, such as the Traveling Salesman Problem (see [**2**]). This approach has the drawback that the computer might not find a solution even when it exists.

In order to use the simulated annealing method we transformed the search for a figure of constant width into an optimization problem. We chose to minimize the objective function

$$\mathcal{E}(F) = \sum_{L} \left|\#(L \cap F) - w\right|,$$

where L runs through all rows, columns, and diagonals on the chessboard that intersect F and $\#(L \cap F)$ signifies the number of squares of F in L. Notice that $\mathcal{E}(F) \geq 0$ and $\mathcal{E}(F) = 0$ if and only if F is a constant width figure.

Figures 12 and 13 show other solutions found with the computer. We have also found constant width figures of types (14, 14, 4) and (17, 17, 5).

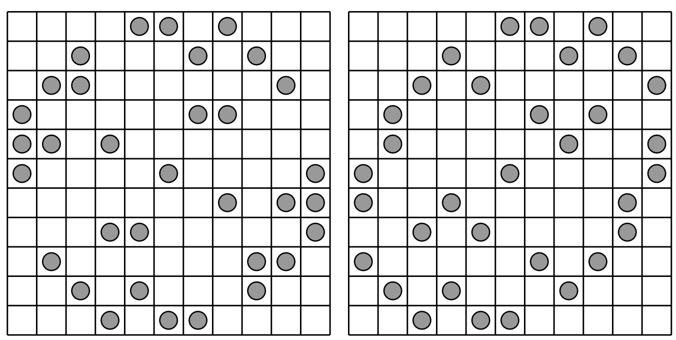

Figure 12. Constant width figures of type (11, 11, 3).

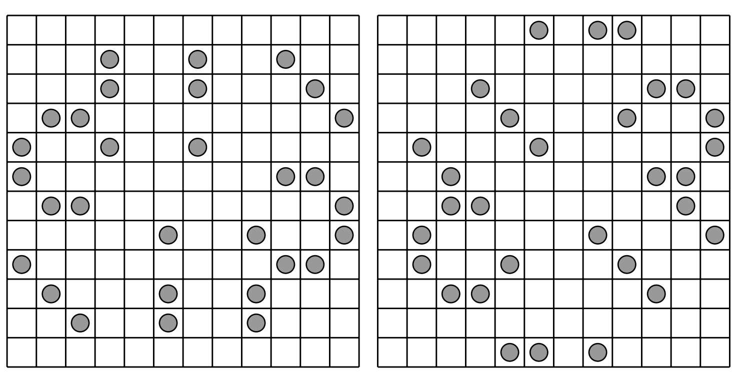

Figure 13. Constant width figures of type (12, 10, 3).

The difficulty in finding constant width figures of type (n, k, w) seems to grow significantly faster with the increase of w than with the increase of n. Our experiences with both backtracking and annealing searches back this conclusion. For example, finding figures of constant width two or three is relatively easy, but we were not able to find a figure of width six with the computer.

We conclude with some conjectures suggested by our computations. Let $W(n, k, w)$ be the number of constant width figures of type (n, k, w). Then we conjecture the following:

1. $W(n, k, 3) = 0$ *if either* (i) $k < 10$ *or* (ii) $n < 11$.
2. $W(11, 10, 3) = 8$.
3. $W(n, k, 4) = 0$ *if either* (i) $n < 14$ *or* (ii) $n = 14$, $k < 14$.
4. *If* $W(n, n, w) > 0$ *for some* n *and* w*, then* $W(m, m, w) > 0$ *whenever* $m \geq n$.

We have been unable to check 1(ii), 2, or 3, because the algorithm that we have used hitherto does not do exhaustive searches. Conjecture 1(i) is of a stronger nature, since it says that there are no constant width figures of type $(n, k, 3)$ if $k < 10$, independent of n. Thus, in principle it cannot be checked by a computer search. Conjecture 2 asserts that the only constant width figures of type (11, 10, 13) are Figure 11 and its rotations and reflections.

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