On a class of Sobolev scalar products in the polynomials.

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Abstract

This paper discusses Sobolev orthogonal polynomials for a class of scalar products that contains the sequentially dominated products introduced by G. L. Lagomasino and H. Pijeira. We prove asymptotics for Markov type functions associated to the Sobolev scalar product and an extension of Widom's Theorem on the location of the zeroes of the orthogonal polynomials. In the case of measures supported in the real line, we obtain results related to the determinacy of the Sobolev moment problem and the completeness of the polynomials in a suitably defined weighted Sobolev space.

Keywords: Sobolev inner product, orthogonal polynomials, asymptotics, location of zeroes, Favard's Theorem, determinacy, completeness.

0 Introduction.

This paper discusses several properties of the sequence of orthonormal polynomials with respect to a Sobolev scalar product of the form

$$\langle p, q\rangle_S = \int_{\Omega_0} p(z)\overline{q(z)}d\mu_0 + \int_{\Omega_1} p'(z)\overline{q'(z)}d\mu_1. \tag{1}$$

where μ_0 and μ_1 are positive Borel measures in the complex plane with supports Ω_0 and Ω_1 respectively.

The analysis is restricted to the class of scalar products that satisfy the condition (5) stated in Section 1. This class includes the sequentially dominated products introduced in [8].

We associate to the sequence of orthonormal polynomials a closed operator $\mathcal{D}$ analogous to the Jacobi operator in the case of the standard orthogonality in the real line. Then from the properties of the scalar product we obtain information about the spectrum of $\mathcal{D}$. This is done in Section 1.

In Section 2 we prove asymptotics and zero location for the orthonormal polynomials (Theorems 4 and 5). These results are derived from the convergence of the finite sections method applied to the operator $\mathcal{D}$. Here we apply the results proven in [10] on Hessenberg matrices.

Section 3 discusses the case when the measures μ_0 and μ_1 are supported in the real line. We relate the concept of determinate Hessenberg matrix introduced in [10] to the Sobolev moment problem. We also prove the density of the polynomials in the weighted Sobolev space associated to the scalar product (1).

Besides [10], [9], our references were [8], [6]. In many ways, this paper continues the work of [8], improving some of its results.

1 Preliminaries.

Let $\langle \cdot, \cdot \rangle : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$ be a scalar product in the linear space $\mathbb{C}[z]$ of polynomials with complex coefficients. Applying the Gram-Schmidt process to the basis $\{z^n\}_{n=0}^{\infty}$ we can find a sequence $\{p_n(z)\}_{n=0}^{\infty}$ of orthonormal polynomials with respect to this scalar product. Since these polynomials form a basis of $\mathbb{C}[z]$, $zp_n(z)$ can be written as a linear combination of $p_i(z)$, $i = 0, \dots, n+1$, for every n . Thus we have a recurrence relation

$$zp_n(z) = \sum_{i=0}^{n+1} d_{n,i} p_i(z).$$

Define the infinite matrix $D = (d_{i,j})_{i,j=0}^{\infty}$. This recurrence relation can be written like

$$D\mathbf{p} = z\mathbf{p}, \tag{2}$$

where $\mathbf{p} = (p_0, p_1, \dots)^t$. Notice that D is a lower Hessenberg matrix; that is, $d_{i,j} = 0$ for $j > i + 1$.

Let l_2 denote the Hilbert space of infinite column vectors with square summable entries and $C_0 \subset l_2$ the subspace of vectors with a finite number of nonzero entries. Associated to the matrix D, we define the operator $\mathcal{D}$ with domain $domain(\mathcal{D}) = \{x \in l_2 : Dx \in l_2\}$ and such that $\mathcal{D}x = Dx$. It is proven in [10] that $\mathcal{D}$ is a closed operator. We denote by $\sigma(\mathcal{D})$, $\rho(\mathcal{D})$, and $R(z, \mathcal{D})$, the spectrum, the resolvent set, and the resolvent function of $\mathcal{D}$, respectively. We use caligraphic fonts to denote the operator associated to a Hessenberg matrix.

Define the set

$$\widetilde{\Gamma}(D) = \overline{\{\langle Dx, x\rangle : \|x\| = 1, x \in C_0\}}.$$

For a vector $x \in C_0$, $x = (x_0, x_1, \dots)^t$, we write $p_x(z) = \sum_i x_i p_i(z)$. In this notation the orthonormal basis $\{p_n(z)\}_{n=0}^{\infty}$ is implicitly assumed, but this will not lead to confusion. From the definition of D we get that $\langle De_n, e_m\rangle = \langle zp_m, p_n\rangle$. Taking linear combinations this yields

$$\langle D\overline{y}, \overline{x}\rangle = \langle zp_x, p_y\rangle \quad x, y \in C_0. \tag{3}$$

Thus, one sees that the set $\widetilde{\Gamma}(D)$ is just the numerical range of the operator $T_z : \mathbb{C}[z] \to \mathbb{C}[z]$, $T_z p = zp$, of multiplication by z.

Example 1. Consider the scalar product

$$\langle p, q\rangle = \int_\Omega p(z)\overline{q(z)}d\mu,$$

where μ is a positive measure supported in $\Omega \subset \mathbb{C}$, with infinite support and finite moments. We have

$$\widetilde{\Gamma}(D) = \overline{\{\lambda \in \mathbb{C} : \lambda = \langle zp, p\rangle,\ \text{with } \|p\| = 1\}}. \quad (4)$$

Let $a \in (Co(\Omega))^c$ where $Co(\Omega)$ is the closed convex hull of Ω. Choose $\omega \in \mathbb{C}$ such that $|\omega| = 1$ and $\Re(\omega(z-a)) > \epsilon > 0$ for every $z \in \Omega$ and let $\|p\| = 1$. Then

$$\begin{aligned} |\langle zp, p\rangle - a| = |\langle (z-a)p, p\rangle| &= \left|\int_\Omega (z-a)|p(z)|^2 d\mu\right| = \\ &= \left|\int_\Omega \omega(z-a)|p(z)|^2 d\mu\right| > \epsilon\|p\|^2 = \epsilon > 0. \end{aligned}$$

This proves that $\widetilde{\Gamma}(D) \subset Co(\Omega)$.

1.1 Sobolev products.

Let μ_0, μ_1 be positive measures in the complex plane with finite moments, and such that at least one of the sets $\Omega_0 = \operatorname{supp}\mu_0$, $\Omega_1 = \operatorname{supp}\mu_1$ is infinite. With these conditions the equation (1) defines a scalar product in $\mathbb{C}[z]$. We will consider additionally that $\int d\mu_0 = 1$, so that one has $\langle 1, 1\rangle_S = 1$. For the rest of this section $\{p_n\}_{n=0}^\infty$ denotes the sequence of orthonormal polynomials with respect to $\langle\cdot,\cdot\rangle_S$ and D denotes the Hessenberg matrix associated to it. We denote by $\|\cdot\|_S$ the norm in $\mathbb{C}[z]$ induced by (1); we write $\|\cdot\|_{S,\mu_0,\mu_1}$ when we want to make explicit reference to the measures μ_0, μ_1.

G. Lagomasino and H. Pijeira introduce in [8] the concept of *sequentially dominated measures*, this being the case when

i) μ_1 is absolutely continuous with respect to μ_0.

ii) $d\mu_1/d\mu_0 \in L_\infty(\mu_0)$.

They base many of their results in this concept, in particular they show that $\mathcal{D}$ is bounded whenever μ_0, μ_1 are sequentially dominated and Ω_0, Ω_1 are compact subsets of the complex plane.

Instead of those assumptions, we will consider here the following condition on μ_0 and μ_1

$$\left(\int_{\Omega_1} |p(z)|^2 d\mu_1\right)^{1/2} = \|p\|_{\mu_1} \le M\|p\|_S \quad (5)$$

for every polynomial $p \in \mathbb{C}[z]$ and some positive constant M.

This condition is equivalent to

$$\|p\|_{S,\mu_0,\mu_1} \le \|p\|_{S,\mu_0+\mu_1,\mu_1} \le (M^2+1)^{1/2}\|p\|_{S,\mu_0,\mu_1}.$$

Thus, (5) can be restated saying that the norms $\|\cdot\|_{S,\mu_0,\mu_1}$ and $\|\cdot\|_{S,\mu_0+\mu_1,\mu_1}$ are equivalent. Notice that the measures $\mu_0+\mu_1$, μ_1 are sequentially dominated.

Sequentially dominated measures satisfy (5), but they are far from being all, as the following example shows.

Example 2. Let $d\mu_1 = \omega dx$ be a positive measure, absolutely continuous with respect to the Lebesgue measure, supported in $[-1,1]$ and such that $1/\omega \in L_1([-1,1],dx)$. Let μ_0 be an arbitrary measure such that $\mu_0([-1,1]) \neq 0$. Let us prove that these measures satisfy condition (5).

Let $p(z) \in \mathbb{C}[z]$ with $\|p\|_S = 1$. Since $\int |p|^2 d\mu_0 \leq 1$, there exists $x_0 \in [-1,1]$ such that $|p(x_0)| \leq (\mu_0([-1,1]))^{-1/2}$. For every $x \in [-1,1]$, we have

$$p(x) = p(x_0) + \int_{x_0}^{x} p'(t)dt,$$

and

$$\begin{aligned}
\left|\int_{x_0}^{x} p'(t)dt\right| &\leq \int_{-1}^{1} |p'(t)|dt = \int_{-1}^{1} |p'(t)| \frac{1}{\omega(t)} d\mu_1 \\
&\leq \left(\int_{-1}^{1} |p'(t)|^2 d\mu_1\right)^{1/2} \left(\int_{-1}^{1} \frac{1}{\omega(t)^2} d\mu_1\right)^{1/2} \leq \left(\int_{-1}^{1} \frac{1}{\omega(t)} dt\right)^{1/2}.
\end{aligned}$$

Thus, $|p(x)| \leq (\mu_0([-1,1]))^{-1/2} + (\int_{-1}^{1} (\omega(t))^{-1} dt)^{1/2}$, and (5) clearly follows from this.

The next theorem gives a description of the set $\widetilde{\Gamma}(D)$ in terms of Ω_0 and Ω_1.

Theorem 1. *Let μ_0, μ_1 satisfy condition (5). Then we have*

i) $\widetilde{\Gamma}(D) \subset \{z : d(z, Co(\Omega_0 \cup \Omega_1)) \leq M\}$.

ii) $\mathcal{D} - zI$ is surjective for $z \in (\Omega_0 \cup \Omega_1)^c$.

iii) If Ω_0 and Ω_1 are bounded sets of the complex plane then $\mathcal{D}$ is a bounded operator.

Proof. Let us write

$$\langle p, q\rangle_0 = \int_{\Omega_0} p\overline{q} d\mu_0, \qquad \langle p, q\rangle_1 = \int_{\Omega_1} p\overline{q} d\mu_1.$$

Taking into account (4), we need to estimate $\langle zp, p\rangle_S$ assuming that $\|p\|_S = 1$.

$$\begin{aligned}
\langle zp, p\rangle_S &= \langle zp, p\rangle_0 + \langle zp', p'\rangle_1 + \int_{\Omega_1} p\overline{p'} d\mu_1 \\
&= \langle p, p\rangle_0 \left(\frac{\langle zp, p\rangle_0}{\langle p, p\rangle_0}\right) + \langle p', p'\rangle_1 \left(\frac{\langle zp', p'\rangle_1}{\langle p', p'\rangle_1}\right) + \int_{\Omega_1} p\overline{p'} d\mu_1.
\end{aligned}$$

Since $\langle p, p\rangle_0 + \langle p', p'\rangle_1 = 1$ the first two summands of the last equality form a convex combination of elements in $\widetilde{\Gamma}(\langle\cdot,\cdot\rangle_0)$ and $\widetilde{\Gamma}(\langle\cdot,\cdot\rangle_1)$. Using (5), the last summand admits the estimate

$$\left|\int_{\Omega_1} p'\overline{p} d\mu_1\right|^2 \leq \int_{\Omega_1} |p|^2 d\mu_1 \int_{\Omega_1} |p'|^2 d\mu_1 \leq M^2.$$

We know from Example 1 that $\widetilde{\Gamma}(\langle\cdot,\cdot\rangle_0) \subset Co(\Omega_0)$ and $\widetilde{\Gamma}(\langle\cdot,\cdot\rangle_1) \subset Co(\Omega_1)$. This completes the proof of i).

ii) Let $z \in (\Omega_0 \cup \Omega_1)^c$ and consider the infinite matrix

$$X(z) = \left(\left\langle \frac{p_i(t)}{z-t}, p_j(t) \right\rangle_t \right)_{i,j}.$$

This matrix satisfies the identity $(zI - D)X(z) = I$. Thus, it is enough to prove that $X(z)$ is a matrix representing a bounded operator of l_2 and ii) will follow from this.

Let $x, y \in C_0$ such that $\|x\| = \|y\| = 1$. We have

$$\begin{aligned} |\langle X(z)x, y\rangle| &= \left| \left\langle \frac{p_x(t)}{z-t}, p_y(t) \right\rangle_t \right| \\ &= \left| \int_{\Omega_0} \frac{p_x(t)}{z-t} \overline{p_y(t)} d\mu_0(t) + \int_{\Omega_1} \left(\frac{p_x'(t)}{z-t} + \frac{p_x(t)}{(z-t)^2} \right) \overline{p_y'(t)} d\mu_1(t) \right| \\ &\leq C \int_{\Omega_0} |p_x \overline{p_y}| d\mu_0 + C \int_{\Omega_1} |p_x' p_y'| d\mu_1 + C^2 \int_{\Omega_1} |p_x p_y'| d\mu_1 \\ &\leq 2C + C^2 M. \end{aligned}$$

iii) If Ω_0 and Ω_1 are bounded then $\widetilde{\Gamma}(D)$ is bounded and by lemma 1 of Section 2, $\mathcal{D}$ is a bounded operator. □

In paper [2] it is proven that if D^k is bounded for some $k > 0$, then Ω_0 and Ω_1 are bounded sets. This in particular implies that the converse of Theorem 1 iii) is also true.

2 Asymptotics and location of the zeroes.

During the last decade several asymptotics for Sobolev orthonormal polynomials were proven. In [8] and [6] the n-th root asymptotics are obtained under the hypothesis of sequentially dominated measures and the regularity of both measures μ_0, μ_1. In these papers the location of the zeroes of the orthogonal polynomials was an important step in the obtention of asymptotics. They also consider Sobolev products involving derivatives of higher order.

In [10], asymptotics for orthogonal polynomials with respect to an arbitrary scalar product were proven exploiting the relation of the polynomials with Hessenberg matrices. More precisely, these asymptotics were obtained based on the applicability of the finite sections method to Hessenberg matrices (Theorems 2 and 3 below). We use the same approach here.

2.1 Finite sections method for Hessenberg matrices.

In this subsection we review some facts about general Hessenberg matrices. We refer to [10] for the proofs of the theorems stated here and for further development.

Let $D = (d_{i,j})_{i,j=0}^{\infty}$ be an infinite lower Hessenberg matrix; that is, $d_{i,j} = 0$ for $j > i+1$ and $d_{i,i+1} \neq 0$ for $i \geq 0$. We associate to D the sequence of polynomials defined by $p_0 = 1$ and the recurrence relation (2). These polynomials form a basis of $\mathbb{C}[z]$, thus there is a unique scalar product $\langle \cdot, \cdot \rangle_D : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$ defined by $\langle p_n, p_m \rangle_D = \delta_{n,m}$.

For example, if J is a real Jacobi matrix, then we know by Favard's Theorem that this scalar product has the form

$$\langle p, q \rangle_J = \int_{-\infty}^{\infty} p(t)\overline{q(t)} d\mu(t),$$

where μ is a positive measure with finite moments and infinite support.

It is known that Stieltjes' Theorem on the convergence of the Padé approximants to the Markov function of μ, is equivalent to the strong convergence of $R(z, J_n)$ to $R(z, \mathcal{J})$ in $z \in \mathbb{C}\backslash\mathbb{R}$. Here J_n are the truncated matrices of the Jacobi matrix J. Theorem 2 below, extends this theorem to Hessenberg matrices. Since Stieltjes' theorem only holds when J is determinate, we need a suitable generalization of this concept.

Let D be Hessenberg matrix and define the sequence of *associated polynomials of k-th kind* (also called *shifted polynomials*) by

$$p_{n-k}^{k}(z) = \left\langle \frac{p_n(z) - p_n(t)}{z - t}, p_{k-1}(t) \right\rangle_{D,t}.$$

In the case of a scalar product in the real line this definition agrees with the standard definition of associated polynomials of k-th kind.

We say that the matrix D is *determinate* if

$$\sum_{n,k=0}^{\infty} |p_{n-k}^{k}(z)|^2 = \infty \tag{6}$$

for at least one $z \in \mathbb{C}$. Again it can be proven that this definition agrees with the standard one for complex Jacobi matrices and a theorem of invariabilty holds (see [13] for the definition of determinate complex Jacobi matrix). In particular, if (6) holds for some $z_0 \in \mathbb{C}$, then it holds for every $z \in \mathbb{C}$. Since $p_0^n(z) = d_{n-1,n}^{-1}$, we have that if $\mathcal{D}$ is bounded, or more generally if $\sum_n |d_{n-1,n}|^{-2} = \infty$, then D is determinate.

Denote by $D_n = (d_{i,j})_{i,j=0}^{n-1}$ the truncated matrix of size $n \times n$ and define the sets

$$\begin{aligned}
\Theta_\Lambda(D) &= \{z : \limsup_{n\in\Lambda} \|R(z, D_n)\| < \infty\}, \\
Z_\Lambda(D) &= \{z : z \in \sigma(D_n), n \in \Lambda\}, \\
Z_\Lambda^\infty(D) &= \{z : \exists\{z_{n_k}\}, z_{n_k} \in \sigma(D_{n_k}), z_{n_k} \to z, n_k \in \Lambda\},
\end{aligned}$$

where $\Lambda \subset \{1, 2, \dots\}$ is an infinite sequence of indices. If $\Lambda = \{1, 2, \dots\}$ we ommit the index and write simply $\Theta(D)$, $Z(D)$, $Z^\infty(D)$. The following inclusions are not hard to proof

$$Z(D) \subset \widetilde{\Gamma}(D), \qquad (\widetilde{\Gamma}(D))^c \subset \Theta(D).$$

If J is a Jacobi matrix we have seen in Section 1 that $\widetilde{\Gamma}(J) \subset Co(\mathrm{supp}\,(\mu)) \subset \mathbb{R}$, hence, $\mathbb{C}\backslash\mathbb{R} \subset \Theta(J)$.

Now we can state the generalization of Stieltjes' Theorem.

Theorem 2. *Suppose D is determinate, then*

$$(\widetilde{\Gamma}(D))^c \subset \Theta(D) = \{z : R(z, D_n)^* \to R(z, \mathcal{D})^*\} \subset \rho(\mathcal{D})\backslash Z^\infty(D). \tag{7}$$

For all $x \in l_2$ we have

$$\lim_{n\in\Lambda} R(z, D_n)^* x = R(z, \mathcal{D})^* x$$

uniformly in compact subsets of $\Theta_\Lambda(D)$.

If $\mathcal{D}$ is bounded, or more generally, if $d_{n-1,n}$ is a bounded sequence, we can improve Theorem 2.

Let $\rho^\infty(\mathcal{D})$ be the union of the connected components of $\rho(\mathcal{D})$ which have nonempty intersection with $(\widetilde{\Gamma}(D))^c$. Notice that since the set $\widetilde{\Gamma}(D)$ is convex, its complement has at most two connected components. We denote by $v(D_n, K)$ the number of eigenvalues of D_n in $K \subset \mathbb{C}$.

Theorem 3. *Suppose that $\widetilde{\Gamma}(D) \neq \mathbb{C}$ and that the sequence $\{d_{n-1,n}\}_{n\in\Lambda}$ is bounded. We have*

i) For every compact set $K \subset \rho^\infty(\mathcal{D})$ there is an infinite subsequence $\Lambda' \subset \Lambda$ such that $K\backslash\Theta_{\Lambda'}(D)$ is at most finite.

ii) $\rho^\infty(\mathcal{D})\backslash Z^\infty_\Lambda(D) \subset \Theta_\Lambda(D)$.

iii) For every $K \subset \rho^\infty(\mathcal{D})$, the sequence $\{v(D_n, K)\}_{n\in\Lambda}$ is uniformly bounded.

Thus, this theorem implies that if $\mathcal{D}$ is bounded and $\rho(\mathcal{D})$ is connected, then $\Theta(D) = \rho(\mathcal{D})\backslash Z^\infty(D)$.

The next lemma lists some conditions equivalent to the boundedness of $\mathcal{D}$.

Lemma 1. *Let D be a Hessenberg matrix with associated scalar product $\langle \cdot, \cdot \rangle_D : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$. The following statements are equivalent:*

i) $|\langle zp, q\rangle_D| \leq C \langle p, p\rangle_D^{\frac{1}{2}} \langle q, q\rangle_D^{\frac{1}{2}}$ for some constant C and every $p, q \in \mathbb{C}[z]$.

ii) $|\langle zp, p\rangle_D| \leq C' \langle p, p\rangle_D$ for some constant C' and every $p \in \mathbb{C}[z]$.

iii) $\widetilde{\Gamma}(D)$ is bounded.

iv) $\mathcal{D}$ is a bounded operator of l_2.

Proof. The implication i)⇒ii) is trivial. We have noticed in Section 1 that $\widetilde{\Gamma}(D)$ is the numerical range of the operator of multiplication by z in $\mathbb{C}[z]$, thus ii)⇔iii). It is a known fact that the boundedness of the numerical range of an operator implies its boundedness, thus ii) implies that the operator of multiplication by z is bounded in $\mathbb{C}[z]$ with respect to the norm induced by the scalar product. Taking into account (3), the rest of the implications follow. □

2.2 Asymptotics.

For the rest of the section $\langle \cdot, \cdot \rangle_S$ is a Sobolev product as in (1) that satisfies (5). We denote by $\{p_n(z)\}_{n=0}^{\infty}$ the orthonormal polynomials and by D the Hessenberg matrix associated to it.

Let us write Ω_∞ for the union of the connected components of $(\Omega_0 \cup \Omega_1)^c$ with nonempty intersection with $(Co(\Omega_0 \cup \Omega_1))^c$ (there can be at most two connected components).

Proposition 1. *We have*

i)

$$\{z : d(z, Co(\Omega_0 \cup \Omega_1)) > M)\} \subset \Theta(D).$$

ii) If D is determinate then $\Omega_\infty \subset \rho^\infty(\mathcal{D})$.

iii) If Ω_0,Ω_1 are bounded or more generally $d_{n-1,n}$ is bounded then

$$\Omega_\infty \backslash Z^\infty(D) \subset \Theta(D) \subset (\Omega_0 \cup \Omega_1)^c \backslash Z^\infty(D).$$

Proof. i) This follows from the inclusion $(\widetilde{\Gamma}(D))^c \subset \Theta(D)$ and Theorem 1 i) of Section 1.

ii) Suppose that D is determinate. By Theorem 2 $(\widetilde{\Gamma}(D))^c \subset \rho(\mathcal{D})$. Hence, Ω_∞ is a connected open set with nonempty intersection with $\rho(\mathcal{D})$ and where $zI - \mathcal{D}$ is surjective (Theorem 1 ii)). It follows that $\Omega_\infty \subset \rho(\mathcal{D})$ and from this that $\Omega_\infty \subset \rho^\infty(\mathcal{D})$.

iii) This follows at once from Theorem 3 ii). □

Lemma 2. *Let $x, y, z \in \rho(\mathcal{D}) \backslash (\Omega_0 \cup \Omega_1)$. We have*

$$\begin{aligned} \left\langle \frac{p_i}{z-t}, p_j \right\rangle &= \langle R(z, \mathcal{D}) e_j, e_i \rangle \\ \left\langle \frac{1}{x-t}, \frac{1}{y-t} \right\rangle &= \langle R(x, \mathcal{D}) R(y, \mathcal{D})^* e_0, e_0 \rangle . \end{aligned}$$

Let $x, y, z \in \rho(D_n)$ then

$$\begin{aligned} \frac{1}{p_n(z)} \left\langle \frac{p_n(z) - p_n(t)}{z-t} p_i, p_j \right\rangle &= \langle R(z, \mathcal{D}_n) e_j, e_i \rangle \\ \left\langle \frac{p_n(x) - p_n(t)}{x-t}, \frac{p_n(y) - p_n(t)}{y-t} \right\rangle &= \langle R(x, \mathcal{D}_n) R(y, \mathcal{D}_n)^* e_0, e_0 \rangle . \end{aligned}$$

These formulas are part of a more general formalism that relates Hessenberg matrices and their finite sections to quadrature formulas, two-variable Padé approximants, and infinite dimensional continued fractions ([9]).

Proof. Recall that the matrix $X(z)$ defined in the proof of Theorem 1 ii) is a bounded right inverse of $zI - \mathcal{D}$. Since now $z \in \rho(D)$, we must have $X(z) = R(z, \mathcal{D})$. This implies the first formula. Define the infinite matrix $(Y(x, \overline{y}))_{i,j} = \langle \frac{p_i}{x-t}, \frac{p_j}{y-t} \rangle$. Analogously as we did for $X(z)$, it can be checked that $Y(x, \overline{y})$ is the matrix of a bounded operator (use condition 5) and satisfies the matrix identity $(xI - D)Y(x,\overline{y})(\overline{yI - D})^t = I$. An analysis of the operators associated to the matrices contained in this identity yields $Y(x,\overline{y}) = R(x,\mathcal{D})R(y,\mathcal{D})^*$, thus we get the second formula.

In order to prove the formulas in the second part of the lemma let us define the $n \times n$ matrices $X_n(z)$ and $Y_n(x,\overline{y})$, whose entries are the left side of these formulas. After some straightforward computations involving the orthogonality relations of the polynomials, one checks that they satisfy $(zI_n - D_n)X_n(z) = I$ and $(xI_n - D_n)Y_n(x,\overline{y})(\overline{yI_n - D_n})^t = I_n$. The last two formulas follow from this. □

The following lemma fills the gap in the proof of the formula for $Y(x,\overline{y})$ in lemma 2.

Lemma 3. *Let D_1, D_2 be Hessenberg matrices such that $\mathcal{D}_1, \mathcal{D}_2$ have bounded inverse and Y is the matrix of a bounded operator $\mathcal{Y}$. Suppose that $D_1 Y \overline{D}_2^t = I$. Then $\mathcal{Y} = \mathcal{D}_1^{-1}(\mathcal{D}_2^{-1})^*$.*

Proof. Denote by D_i^{-1}, $i = 1,2$, the matrices of the bounded operators $\mathcal{D}_i^{-1}$. Recall that taking the adjoint of a bounded operator corresponds to taking the conjugate transpose of its matrix. Let $x \in C_0$, then $(D_2)^t x \in C_0$ and $y = Y\overline{D}_2^t x$ is well defined since $\mathcal{Y}$ is bounded. We have $y \in l_2$ and $D_1 y = x$, thus $y \in domain(\mathcal{D}_1)$ and $y = D_1^{-1}x$. That is, $Y\overline{D}_2^t x = D_1^{-1}x$, for every $x \in C_0$. This implies that $Y\overline{D}_2^t = D_1^{-1}$. Taking conjugate transpose and repeating the same analysis we get $\overline{Y}^t = \overline{D_1^{-1}}^t D_2^{-1}$. □

Theorem 4. *Let $\langle \cdot,\cdot \rangle_S$ be a Sobolev scalar product with determinate Hessenberg matrix. We have*

$$\frac{1}{p_n(z)} \int \frac{p_n(z) - p_n(t)}{z - t} d\mu_0 \quad \to \quad \int \frac{1}{z-t} d\mu_0,$$
$$\frac{1}{p_n(z)} \int \left(\frac{p_n(z) - p_n(t)}{z - t}\right)' d\mu_1 \quad \to \quad -\int \frac{1}{(z-t)^2} d\mu_1$$

uniformly in compact subsets of $\Theta(D)$ *and*

$$\frac{1}{p_n(x)\overline{p_n(y)}} \int \left(\frac{p_n(x) - p_n(t)}{x - t}\right)' \overline{\left(\frac{p_n(y) - p_n(t)}{y - t}\right)'} d\mu_1 \to \int \frac{1}{(x-t)^2\overline{(y-t)^2}} d\mu_1$$

uniformly in compact subsets of $\Theta(D) \times \Theta(D)$.

Under the conditions stated in Theorem 3, the same is true for subsequences of indices.

Proof. The first two limits in the statement of the theorem follow taking $i = 0, j = 0,1$ in the formulas of lemma 2 and applying Theorem 2. The third limit follows readily from lemma 2 and Theorem 2. □

2.3 Location of zeroes.

Just like in the case of orthogonal polynomials in the real line, it can be checked that we have (see [8] for a proof):

$$\sigma(D_n) = \{z \in \mathbb{C} : p_n(z) = 0\}. \tag{8}$$

We see from this formula that the boundedness of $\mathcal{D}$ implies the boundedness of the zeroes of $p_n(z)$, because $\|D_n\| \leq \|\mathcal{D}\|$ and the eigenvalues of D_n are contained in the disk of radius $\|D_n\|$ centered in the origin. Alternatively, we can argue that if $p_n(z) = (z-a)q(z)$, then

$$\langle (z-a)q, q\rangle = 0 \Rightarrow \langle zq, q\rangle = a\langle q, q\rangle$$

and from Lemma 1 ii) we get $|a| < C'$.

The boundedness of the zeroes of the orthonormal polynomials does not imply the boundedness of $\mathcal{D}$. We will discuss this phenomenon in Section 3.

By (8), the sets $Z_\Lambda(D)$, $Z_\Lambda^\infty(D)$ defined in 2.1 can be reinterpreted in terms of the zeroes of the orthogonal polynomials. Theorem 1 together with the fact that $Z(D) \subset \widetilde{\Gamma}(D)$ implies that

$$Z(D) \subset \{z : d(z, Co(\Omega_0 \cup \Omega_1)) \leq M\}.$$

The following theorem generalizes a well known theorem by H. Widom on the behaviour of the zeroes of orthonormal polynomials.

Theorem 5. *Let $\langle \cdot, \cdot \rangle_S$ be a Sobolev scalar product satisfying condition (5). Suppose that $Co(\Omega_0 \cup \Omega_1) \neq \mathbb{C}$ and $\{d_{n-1,n}\}_{n\in\Lambda}$ is bounded for some subsequence $\Lambda \subset \{0, 1, \dots\}$. Then for every compact $K \subset \Omega_\infty$ the sequence $\{v(K, p_n)\}_{n\in\Lambda}$ is bounded. In particular this is true when Ω_0, Ω_1 are bounded and Ω_∞ is the unbounded connected component of $(\Omega_0 \cup \Omega_1)^c$. In this case Λ can be taken to be $\{0, 1, 2, \dots\}$.*

Taking $\mu_1 = 0$ condition (5) is automatically statisfied and we get an extension of Widom's Theorem to measures of unbounded support.

Proof. Since $\{d_{n-1,n}\}_\Lambda$ is bounded D is determinate. By Theorem 1 i), if $Co(\Omega_0 \cup \Omega_1) \neq \mathbb{C}$ then $\widetilde{\Gamma}(D) \neq \mathbb{C}$. Combining Theorem 1 ii) and Theorem 3 iii) we get the first part of the theorem. If Ω_0 and Ω_1 are bounded then $\mathcal{D}$ is bounded, thus $\{d_{n-1,n}\}_{n=0}^\infty$ is bounded. □

3 Sobolev products in the real line.

In this section we assume that the measures μ_0 and μ_1 are supported in the real line. Now the scalar product is

$$\langle p, q\rangle_S = \int_{-\infty}^{\infty} p(t)\overline{q(t)}d\mu_0(t) + \int_{-\infty}^{\infty} p'(t)\overline{q'(t)}d\mu_1(t). \tag{9}$$

3.1 Formal properties.

The restriction $\Omega_0, \Omega_1 \subset \mathbb{R}$ induces some formal properties in the scalar product. Let us see how.

Let us associate to every Hermitian bilinear form $\{\cdot,\cdot\} : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}$, a linear functional $\Lambda : \mathbb{C}[x, \overline{y}] \to \mathbb{C}$ by

$$\Lambda(p(x)\overline{q(y)}) = \{p, q\}. \tag{10}$$

If the bilinear form in the right side of (10) is of the form

$$\{p, q\} = \int_{\mathbb{R}} p(t)\overline{q(t)} d\mu(t), \tag{11}$$

with μ a complex measure with finite moments (i.e. $z^n \in L_1(|\mu|)$), then Λ satisfies $\Lambda((x - \overline{y})(\cdot)) = 0$. Conversely, it is proven in [12] that if Λ satisfies $\Lambda((x - \overline{y})(\cdot)) = 0$ then it can be represented by a complex measure μ like in (11). For bilinear forms like (9), we have the next theorem.

Theorem 6. *Let $\{\cdot,\cdot\}_S$ be a Hermitian bilinear form and Λ its associated linear functional defined as in (10). Then $\Lambda((x - \overline{y})^3 p(x, \overline{y})) = 0$ if and only if $\{\cdot,\cdot\}_S$ has the form (9) with μ_0, μ_1 complex measures with finite moments.*

Proof. Suppose that $\{\cdot,\cdot\}$ is like (9). Then

$$\Lambda(p(x, \overline{y})) = \int_{\Omega_0} p(t, t) d\mu_0 + \int_{\Omega_1} \frac{\partial^2 p}{\partial x \partial \overline{y}}(t, t) d\mu_1, \tag{12}$$

and thus $\Lambda((x - \overline{y})^3 p(x, \overline{y})) = 0$. It also follows that

$$\Lambda((x - \overline{y})^2 p(x, \overline{y})) = -2 \int_{\Omega_1} p(t, t) d\mu_1.$$

Conversely if Λ anhilates at the multiples of $(x - \overline{y})^3$ then it can be checked that the linear functionals

$$\Lambda_1(p(x, \overline{y})) = -\frac{1}{2}\Lambda((x - \overline{y})^2 p(x, \overline{y})), \tag{13}$$

$$\Lambda_0(p(x, \overline{y})) = \Lambda(p(x, \overline{y})) - \Lambda_1\left(\frac{\partial^2 p(x, \overline{y})}{\partial x \partial \overline{y}}\right), \tag{14}$$

both anhilate at multiples of $(x - \overline{y})$. Thus, they have an integral representation of the form (11), with μ_0, μ_1 complex measures of finite moments. Therefore, a representation like (12) holds. □

Define the matrix of moments of the bilinear form $\{\cdot,\cdot\}$ (or of the functional Λ) as $(M)_{i,j} = \{z^i, z^j\} = \Lambda(x^i \overline{y}^j)$. Then,

$$\Lambda((x - \overline{y})^3 p(x, \overline{y})) = 0 \iff (S^*)^3 M - 3(S^*)^2 M S + 3 S^* M S^2 - M S^3 = 0, \tag{15}$$

where the right side is understood as an identity of infinite matrices and $(S)_{i,j} = \delta_{i+1,j}$ is the infinite shift matrix. Hence, (15) characterizes the moment matrices of bilinear forms of the form (9) with μ_0, μ_1 complex measures.

When $\{\cdot,\cdot\}$ is a scalar product, we have seen in Section 1 how to associate a Hessenberg matrix to it. It can be proven that the matrix D is related to the functional Λ through the identity (see [9]):

$$\Lambda(p(x,\overline{y})p_i(x)\overline{p_j(y)}) = \langle P(D,\overline{D}^t)e_i, e_j\rangle. \tag{16}$$

Using this equality we get that (15) is equivalent to

$$D^3 - 3D^2\overline{D}^t + 3D(\overline{D}^t)^2 - (\overline{D}^t)^3 = 0. \tag{17}$$

Notice that the entries of D are the coefficients of a recurrence relation for the orthonormal polynomials. So the last identity can be understood like a Favard's theorem for Sobolev products, since it characterizes the Hessenberg matrix of scalar products of the form (9). Notice that in the scalar products of the form (9) we do not assume that μ_0 and μ_1 are positive meaures; we only require that they induce a positive scalar product.

The operators associated to infinite matrices M satisfying (15) have been studied in papers such as [3], where they are called Hankel operators of third order. The decomposition of Λ in the sum of Λ_0 and Λ_1 can be translated in terms of moment matrices. Notice that the moment matrices of Λ_0, Λ_1 will be standard Hankel matrices. This decomposition is proven in [1] and it is used to study the moment problem of Sobolev scalar products.

3.2 The condition (5).

In the sequel we assume that $\langle p,q\rangle_S$ is a Sobolev product of the form (9) with μ_0, μ_1 positive measures supported in the real line. As before, we denote by $\{p_n\}_{n=0}^{\infty}$ the sequence of orthonormal polynomials and D the associated Hessenberg matrix. Notice that since $\langle p,q\rangle_S \in \mathbb{R}$ for every $p, q \in \mathbb{R}[z]$, the coefficients of $p_n(z)$ and the entries of D are real numbers.

For an arbitrary Sobolev product the condition (5) can be understood in terms of the moments of the functionals $\int \cdot d\mu_0$ and $\int \cdot d\mu_1$. But for Sobolev products in the real line, (5) can be put in terms of the moments of the scalar product itself. The next theorem shows equivalent formulations of (5) that hold in this case.

Theorem 7. *The following statements are equivalent:*

i) $\langle\cdot,\cdot\rangle_S$ satisfies (5).

ii) If we write $D = D_r + D_i$ with $D_r = D_r^t$ and $D_i = -D_i^t$ then D_i and DD_i are both matrices of a bounded operator.

iii) $D^2 - 2DD^t + (D^t)^2$ is the matrix of a bounded operator.

This theorem shows how restrictive (5) is, still it doesn't exclude cases where $\mathcal{D}$ is unbounded.

Proof. i)⇒ii).

Since $D_i = \frac{1}{2}(D - D^t)$, we have

$$(D_i)_{n,m} = \frac{1}{2}(\langle zp_n, p_m\rangle - \langle p_n, zp_m\rangle) = \frac{1}{2}\int_{\Omega_1}(p_n p_m' - p_n' p_m)d\mu_1,$$

and thus for $p_u = \sum_i u_i p_i$, $p_v = \sum_i v_i p_i$, with $u, v \in C_0$

$$\langle D_i u, v\rangle = \frac{1}{2}\int_{\Omega_1}(p_u p_v' - p_u' p_v)d\mu_1. \tag{18}$$

Using (5) in the right side of (18) we see that D_i is bounded. Taking $p_v(z) = zp_u(z)$ we get

$$\langle DD_i u, u\rangle = \langle D_i u, D^t u\rangle = \frac{1}{2}\int p_u^2 d\mu_1,$$

and again by (5) DD_i represents a bounded operator too.

ii)⇒iii).

$D^2 - 2DD^t + (D^t)^2 = 2(DD_i - D_i D^t)$. DD_i is bounded by assumption and $D_i D^t = -(DD_i)^t$, so $D_i D^t$ is bounded too.

iii)⇒i).

Let us put $p_u(z) = \sum_i u_i p_i(z)$. It follows from (13) and (16) that

$$\begin{aligned}\int p_u^2 d\mu_1 &= \Lambda((x-\overline{y})^2 p_u(x)\overline{p_u(y)})\\ &= \langle (D^2 - 2DD^t + (D^t)^2)u, u\rangle \leq M\|u\|^2 = M\langle p_u, p_u\rangle.\end{aligned}$$

□

Theorem 8. *$\mathcal{D}$ is bounded if and only if the measures μ_0 and μ_1 have bounded support and satisfy (5).*

This theorem was first proven by J. M. Rodríguez in [11] under the assumption that Ω_0, Ω_1 are bounded sets. This assumption was removed in paper [2]. Papers [2] and [11] consider Sobolev products involving derivatives of arbitrary order; in [2] the measures are supported in subsets of the complex plane. The proof given below is independent of these results.

Proof. Theorem 1 iii) is one of the implications. Suppose that $\mathcal{D}$ is bounded. By Theorem 7 iii) the measures μ_0, μ_1 satisfy (5). Since $\mathcal{D}$ is bounded D is determinate, thus by lemma 2 we have

$$\begin{aligned}\langle R(z,\mathcal{D})e_0, e_0\rangle &= \int \frac{1}{z-t}d\mu_0(t)\\ \langle R(z,\mathcal{D})e_0, e_1\rangle &= \left\langle \frac{1}{z-t}, p_1(t)\right\rangle_S = \int \frac{p_1(t)}{z-t}d\mu_0 - \frac{1}{d_{0,1}}\int \frac{1}{(z-t)^2}d\mu_1.\end{aligned}$$

This implies that the Cauchy transforms of μ_0 and μ_1 are both analytic in a neighborhood of ∞ and therefore, they have bounded support (use Stieltjes' inversion formula). □

We will not discuss the asymptotics for Sobolev orthogonal polynomials in the real line satisfying (5). They are obtained as corollaries of the ones discussed in the previous section. Notice that now we have

$$\widetilde{\Gamma}(D) \subset \{z \in \mathbb{C} : |\Im z| \le M\},$$

and if D is determinate $\mathbb{C}\backslash\mathbb{R} \subset \rho(\mathcal{D})$. Therefore, all of the asymptotics apply for $\{z \in \mathbb{C} : |\Im z| > M\}$ whenever $\mathcal{D}$ is determinate (even though it could be unbounded). We also have $\Omega_\infty = (\Omega_0 \cup \Omega_1)^c$, so if Ω_0 and Ω_1 are bounded, or more generally if $d_{n-1,n}$ is bounded, then the same asymptotics hold outside the set $\Omega_0 \cup \Omega_1 \cup Z^\infty(D)$.

3.3 Location of zeroes.

For a standard scalar product in the real line like (11), it is well known that the boundedness of the set $Z(J)$ of zeroes of the orthonormal polynomials implies the boundedness of the Jacobi operator $\mathcal{J}$. As we mentioned in Section 2, this is not true for general scalar products. For Sobolev products in the real line one can prove the following theorem, which answers a question posed in [7].

Theorem 9. *If a Sobolev product like (9) satisfies (5), then the boundedness of $Z_\Lambda(D)$ for some infinite sequence of indices Λ is equivalent to the boundedness of $\mathcal{D}$.*

Proof. The implication "$\mathcal{D}$ bounded" $\Rightarrow$ "$Z_\Lambda(D)$ bounded" has been discussed already. In order to prove the converse we use the following proposition (see Corollary 6.3.4, [5]).

Proposition 2. *Let A, Δ be $n \times n$ matrices with A normal and $\|\Delta\| = \delta$. Then the spectrum of $A + \Delta$ lies in $\cup_{i=1}^n B_\delta(\lambda_i)$, where λ_i are the eigenvalues of A and $B_\delta(\lambda_i)$ are open balls centered at λ_i of radius δ. If these balls are disjoint, there is at least one eigenvalue of $A + \Delta$ in each one of them.*

If the Sobolev scalar product $\langle \cdot, \cdot \rangle_S$ satisfies (5), then by Theorem 7 the matrices D_n are perturbations of selfadjoint matrices $(D_r)_n$ by a sequence of matrices of uniformly bounded norm $(D_i)_n$. Proposition 2 yields the implication $Z_\Lambda(D)$ bounded $\Rightarrow$ $Z(D_r)$ bounded (prove the contrapositive). But since $(D_r)_n$ is selfadjoint, its norm is equal to its spectral radius, thus, the sequence $\{\|(D_r)_n\|\}_{n\in\Lambda}$ is uniformly bounded. This in turn implies that D_r is the matrix of a bounded operator, and in virtue of Theorem 7 ii), the same is true for D. $\square$

What happens if we drop condition (5)? Then Theorem 9 no longer holds.

Example 3. Let

$$\langle p, q \rangle = p(a)q(a) + \int_{-1}^{1} p'(t)q'(t)dt,$$

where dt denotes the Lebesgue measure. The orthonormal polynomials with respect to this product are obviously $\hat{l}_{n+1}(z) = \int_a^z l_n(t)dt$ where $l_n(t)$ are the Legendre polynomials. We choose $a \notin [-1,1]$ in order to violate (5) (recall Example 2 of Section 1). The set of zeroes of the polynomials $\{\hat{l}_n(z)\}_{n=0}^{\infty}$ is bounded. This is a consequence of the following theorem, proven in [8].

Theorem 10. *Suppose that $Co(\Omega_0)$ and $Co(\Omega_1)$ are disjoint. Then the zeroes of p_n' are simple and contained in $Co(\Omega_0 \cup \Omega_1)$ and the zeroes of p_n lie in the disk centered at the extreme point of $Co(\Omega_1)$ furthest away from Ω_0, and of radius equal to twice the diameter of $Co(\Omega_0 \cup \Omega_1)$.*

Some calculations show that

$$|e_n^t D_n|^2 = \langle z\hat{l}_n, z\hat{l}_n \rangle = \int_{-1}^{1} t^2 l_{n-1}^2(t)dt + \hat{l}_n(1)^2 + \hat{l}_n(-1)^2.$$

Since the right side of this equality is unbounded it follows that $\mathcal{D}$ cannot be bounded.

3.4 Determinacy and completeness.

In [1] a Sobolev scalar product is said to be determinate if there is a unique pair of positive measures μ_0, μ_1 such that (1) holds. The authors notice that, since the functionals Λ_0 and Λ_1 in (13) can be put in terms of Λ (and the same for their respective moment matrices), determinacy holds if and only if both Λ_0 and Λ_1 are determinate. On the other hand, the definition of determinate Hessenberg matrix has been given in Section 2.

Theorem 11. *Suppose a Sobolev scalar product in the real line satisfies (5). Consider the following propositions:*

i) The scalar product is determinate (as defined in [1]).

ii) $\sum_{n=0}^{\infty} |p_n(z_0)|^2 = \infty$ for some $z_0 \in \mathbb{C}\backslash\mathbb{R}$.

iii) The associated Hessenberg matrix is determinate (as defined in Section 2).

iv)

$$\sum_{n=0}^{\infty} |d_{n-1,n}|^{-2} = \infty \quad or \quad \sum_{n=0}^{\infty} |\frac{d_{n,n}}{d_{n-1,n}d_{n,n+1}}|^2 = \infty.$$

We have

$$iv) \Rightarrow iii) \Leftrightarrow ii) \Rightarrow i).$$

Proof. iv)⇒iii). Because $p_0^n(z) = d_{n-1,n}^{-1}$ and $p_1^n(z) = \frac{z-d_{n,n}}{d_{n-1,n}d_{n,n+1}}$.

iii)⇒ii). When D is determinate we have already noticed that $\sigma(\mathcal{D}) \subset (\Omega_0 \cup \Omega_1) \subset \mathbb{R}$. In particular, the point spectrum of $\mathcal{D}$ is contained in $\mathbb{R}$.

ii)⇒iii). If D is indeterminate then

$$\sum_{n,k} |p_{n-k}^k(z)|^2 < \infty,$$

for every $z \in \mathbb{C}$. Since $p_n^0(z) = p_n(z)$, the implication follows.

iii)$\Rightarrow$i). If D is determinate then the asymptotics of Theorem 4 hold in the set $\{z \in \mathbb{C} : \Im z \geq M\}$. Thus, the Cauchy transforms of μ_0 and μ_1 are uniquely determined by the moments of the scalar product and so are both measures. $\square$

Let us define $L_2^{(1)}(\mu_0, \mu_1) = \{f \in C^1(\mathbb{R}) : f \in L_2(\mu_0), f' \in L_2(\mu_1)\}$. For every $f, g \in L_2^{(1)}(\mu_0, \mu_1)$, the scalar product $\langle f, g \rangle_S$ given by (9) is well defined. Let $W_2^1(\mu_0, \mu_1)$ be the completion of $L_2^{(1)}(\mu_0, \mu_1)$ with respect to the norm $\|\cdot\|_S$. If $\mu_1 = 0$ we put $W_2^1(\mu_0, 0) = L_2(\mu_0)$.

Proposition 3. *The linear subspace $\mathbb{C}[x] \oplus span\{\frac{1}{x-a} : a \in \mathbb{C}\backslash\mathbb{R}\}$ is dense in $W_2^1(\mu_0, \mu_1)$.*

Proof. Let us denote by H the closure of $\mathbb{C}[x] \oplus span\{\frac{1}{x-a} : a \in \mathbb{C}\backslash\mathbb{R}\}$ in $W_2^1(\mu_0, \mu_1)$.

For every $a \in \mathbb{C}\backslash\mathbb{R}$, $k \in \mathbb{N}$, there are $a_i \in \mathbb{C}\backslash\mathbb{R}$ distinct, but close enough to a, such that

$$\left|\frac{1}{(x-a)^k} - \frac{1}{\Pi_{i=1}^k (x-a_i)}\right| < \epsilon, \quad \left|\left(\frac{1}{(x-a)^k}\right)' - \left(\frac{1}{\Pi_{i=1}^k (x-a_i)}\right)'\right| < \epsilon$$

for every $x \in \mathbb{R}$. Taking linear combinations, we see that H contains all rational functions with poles off the real line.

Before continuing with the proof of Proposition 3 we need the following lemma.

Lemma 4. *Let $f \in C^1(\mathbb{R})$ that satisfies*

1. $\lim_{|x|\to\infty} x f(x) = 0$.
2. $\lim_{|x|\to\infty} x^3 f'(x) = 0$.

For every $\epsilon > 0$, there is a rational function $r(x)$ with poles off the real line such that such that $|f(x) - r(x)| < \epsilon(1 + |x|)$ and $|f'(x) - r'(x)| < \epsilon(1 + |x|)$ for all $x \in \mathbb{R}$.

Proof. The proof follows the idea of Theorem II.4.2 in [4].

Let $f \in C^1(\mathbb{R})$ be as in the statement of the lemma and suppose additionally that

3. $f(x) = 0$ for $-1 \leq x \leq 1$.
4. $f(x) = f(-x)$.

By 4., there is $g \in C^1(0, 1)$ such that $f(x) = g(\frac{1}{x^2+1})$. Conditions 1-3 imply that $g \in C^1[0, 1]$ (this can be checked after some straightforward computations). Choose a polynomial $p(x)$ such that $|g(x) - p(x)| < \epsilon/2$ and $|g'(x) - p'(x)| < \epsilon/2$

for all $x \in [0,1]$ (e.g. use Bernstein polynomials). We have

$$\left| f(x) - p\left(\frac{1}{x^2+1}\right) \right| < \epsilon/2, \tag{19}$$

$$\left| f'(x) - \left(p\left(\frac{1}{x^2+1}\right)\right)' \right| = \left|\frac{2x}{(x^2+1)^2}\right| \left| g'\left(\frac{1}{x^2+1}\right) - p'\left(\frac{1}{x^2+1}\right)\right| < \frac{\epsilon}{2}. \tag{20}$$

Taking $r(x) = p(1/(x^2+1))$ proves the lemma for $f \in C^1(\mathbb{R})$ that satisfies conditions 1-4.

Let $f \in C^1(\mathbb{R})$ satisfy 1-3. Write $f(x) = f_1(x) + x f_2(x)$ with $f_1(x) = (f(x) + f(-x))/2$ and $f_2(x) = (f(x) - f(-x))/(2x)$. Functions f_1 and f_2 both satisfy conditions 1-4 (this can be checked after some straightforward computations), so we can approximate f_1 and f_2 with rational functions r_1, r_2 like in (19) and (20). Taking $r(x) = r_1(x) + x r_2(x)$ we prove the lemma for $f \in C^1(\mathbb{R})$ that satisfies conditions 1-3.

Let f be as in the statement of the lemma. Using a partition of unity we can write $f = f_1 + f_2$ with $f_1, f_2 \in C^1(\mathbb{R})$ such that $\operatorname{supp} f_1 \subset [1,\infty)$ and $\operatorname{supp} f_2 \subset (-\infty, 2]$. Functions f_1 and $f_2(-x+3)$ satisfy conditions 1-3, thus the lemma is true for each of them. This in turn implies that the lemma holds for f_1 and f_2 and, therefore, the lemma holds for f too. □

Now we continue with the proof of Proposition 3.

Let $f \in L_2^{(1)}(\mu_0, \mu_1)$ such that $f(x) = O(x^n)$, $f'(x) = O(x^n)$ for some $n \geq 0$. Then $f(x)/(x^{2n+4}+1)$ satisfies properties 1. and 2. of the last lemma. Applying the lemma to $f(x)/(x^{2n+4}+1)$ we conclude that $f \in H$.

Let $f \in L_2^{(1)}(\mu_0, \mu_1)$. Using a partition of unity we can write $f = f_1 + f_2$ with $f_1, f_2 \in L_2^{(1)}(\mu_0, \mu_1)$ such that $\operatorname{supp} f_1 \subset [0, \infty)$ and $\operatorname{supp} f_2 \subset (-\infty, 1]$. It is enough to prove that $f_1, f_2 \in H$.

In what follows we assume that $\operatorname{supp} f \subset [0, \infty)$, the analysis for the case $\operatorname{supp} f \subset (-\infty, 1]$ is analogous. We have two possibilities, either

i) $\exists \{a_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} |f(a_n)| = l < \infty$,

or

ii) $\exists \{a_n\}_{n=0}^{\infty}$ such that $\lim_{n\to+\infty} a_n = \infty$ and $|f(x)| \geq |f(a_n)|$ for $x \geq a_n$.

Let $\epsilon_n > 0$ such that $|f(x)| < |f(a_n)| + 1$ for $x \in [a_n, a_n + \epsilon_n]$. Take $h_n \in C^\infty(\mathbb{R})$ such that $0 \leq h_n \leq 1$, $h_n(x) = 1$ for $x \in [0, a_n]$, $h_n(x) = 0$ for $x \in [-\epsilon_n, a_n + \epsilon_n]^c$ and $h_n'(x) \leq 0$ for $x \geq 0$. We approximate f in the Sobolev norm by the function $g_n(x) = \int_0^x f'(t) h_n(t) dt$.

By integration by parts we have

$$g_n(x) = \int_0^x f'(t) h_n(t) dt = f(x) h_n(x) - \int_0^x f(t) h_n'(t) dt.$$

Let us write $r_n(x) = \int_0^x f(t) h_n'(t) dt$. We have that $f(x) h_n'(x) = 0$ for

$x \in [a_n, a_n+\epsilon_n]^c$; thus we get that $r_n(x) = 0$ for $x \leq a_n$ and

$$|r_n(x)| \leq -\int_{a_n}^{a_n+\epsilon_n} |f(t)|h_n'(t)dt \leq \sup_{(a_n,a_n+\epsilon_n)} |f| < |f(a_n)|+1, \quad \text{for } x \geq a_n,$$

hence $r_n(x) = O(1)$.

Since $r_n(x) = O(1)$, $r_n'(x) = O(1)$, and $f(x)h_n(x)$ has compact support, we conclude that $g_n \in H$. We also have

$$\begin{aligned} f(x) - g_n(x) &= f(x)(1-h_n(x)) + r_n(x), \\ f'(x) - g_n'(x) &= f'(x)(1-h_n(x)). \end{aligned}$$

By Lebesgue's dominated convergence theorem $f(x)(1-h_n(x))$ and $f'(x)(1-h_n(x))$ tend to zero in $L_2(\mu_0)$ and $L_2(\mu_1)$ respectively when $n \to \infty$. It only remains to prove that $r_n(x)$ tends to zero in $L_2(\mu_0)$ when $n \to \infty$.

If i) holds, then

$$\int_{-\infty}^{\infty} |r_n(x)|^2 d\mu_0 = \int_{a_n}^{\infty} |r_n(x)|^2 d\mu_0 < (|f(a_n)|+1)^2 \mu_0([a_n,\infty)) \to 0$$

when $n \to \infty$.

If ii) holds, then

$$\begin{aligned} \int_{-\infty}^{\infty} |r_n(x)|^2 d\mu_0 = \int_{a_n}^{\infty} |r_n(x)|^2 d\mu_0 &< \int_{a_n}^{\infty} (|f(a_n)|+1)^2 d\mu_0 \\ &\leq \int_{a_n}^{\infty} (|f(x)|+1)^2 d\mu_0 \to 0 \end{aligned}$$

when $n \to \infty$. This completes the proof of Proposition 3. □

Theorem 12. *Suppose a Sobolev scalar product in the real line satisfies (5) and has determinate Hessenberg matrix. Then the system $\{p_n(z)\}_{n=0}^{\infty}$ of orthonormal polynomials is complete in $W_2^1(\mu_0,\mu_1)$.*

The determinacy of D is not necessary. Even in the case of real Jacobi matrices (i.e. $\mu_1 = 0$), completeness of the polynomials is known to hold for the Von Neumann solutions of the moment problem.

Proof. Let us denote by H the closure of $\mathbb{C}[x]$ in $W_2^1(\mu_0,\mu_1)$. We have proven in lemma 2 of Section 2 that

$$\left\|\frac{1}{x-a}\right\|_S^2 = \|R(x,\mathcal{D})e_0\|^2 = \sum_{i=0}^{\infty} |\langle R(x,\mathcal{D})e_0, e_i\rangle|^2 = \sum_{i=0}^{\infty} \left|\left\langle \frac{1}{x-a}, p_i\right\rangle\right|^2$$

for $a \in (\Omega_0 \cup \Omega_1)^c$. This is Parseval's equality, thus $\frac{1}{x-a} \in H$ for $a \in \mathbb{C}\backslash\mathbb{R}$. Now the theorem follows from Propostion 3. □

We end this section discussing an embedding of $W_2^1(\mu_0,\mu_1)$ in a space of functions.

Let us write $\mu = \mu_0 + \mu_1$, and define the operator $\mathcal{T} : \mathbb{C}[z] \to L_2(\mu)$, $\mathcal{T}p = p$. Then, condition (5) just says that $\mathcal{T}$ is bounded. If we assume that the Hessenberg matrix of the Sobolev product is determinate, by the completeness of the polynomials, the operator $\mathcal{T}$ is uniquely extended to a bounded operator in $W_2^1(\mu_0,\mu_1)$.

When is $\mathcal{T} : W_2^1(\mu_0,\mu_1) \to L_2(\mu)$ injective? Assume that $\mathcal{T}x = 0$, $x \neq 0$. Then there is a sequence of polynomials $q_n \to x$ in $W_2^1(\mu_0,\mu_1)$ and $\|q_n\|_{2,\mu} \to 0$. Since $\{q_n\}_{n=0}^{\infty}$ is convergent in $W_2^1(\mu_0,\mu_1)$, $\{q_n'\}_{n=0}^{\infty}$ is a Cauchy sequence in $L_2(\mu_1)$, hence it is convergent to some $f \in L_2(\mu_1)$. We have $\|f\|_{2,\mu_1} = \lim \|q_n'\|_{2,\mu_1} = \lim \|q_n\|_S = \|x\|_S$, thus $f \neq 0$.

We have obtained that $q_n \to 0$ in $L_2(\mu)$ and $q_n' \to f \neq 0$ in $L_2(\mu_1)$. This means that the operator $\frac{d}{dt} : L_2(\mu) \to L_2(\mu_1)$ with domain $\mathbb{C}[x]$ is not closable.

Suppose that $\frac{d}{dt} : L_2(\mu) \to L_2(\mu_1)$ with domain $\mathbb{C}[x]$ is not closable. Then, there is a sequence of polynomials $\{q_n\}_{n=0}^{\infty}$ such that $q_n \to 0$ in $L_2(\mu)$ and $q_n' \to f \neq 0$ in $L_2(\mu_1)$. $\{q_n\}$ and $\{q_n'\}$ are Cauchy sequences in $L_2(\mu_0)$ and $L_2(\mu_1)$ respectively. Hence, $\{q_n\}$ converges to some $x \in W_2^1(\mu_0,\mu_1)$ in the Sobolev norm and $\|x\|_S = \|f\|_{2,\mu_1} \neq 0$. We have obtained the following:

Proposition 4. *The operator $\mathcal{T} : W_2^1(\mu_0,\mu_1) \to L_2(\mu)$ is injective if and only if $\frac{d}{dt} : L_2(\mu) \to L_2(\mu_1)$ is closable.*

4 Conclusions.

This paper exemplifies a procedure to study scalar products in the linear space of polynomials. This approach is particularly useful for less "standard" products other than in the real line or the circle. The specific properties of a scalar product are used to derive spectral information about the Hessenberg matrix associated to it. In that process, we focus on the computation of sets like $\widetilde{\Gamma}(D)$, $\rho(\mathcal{D})$ or less amenable sets like $\Theta(D)$, $Z^{\infty}(D)$. Then we use the relation between the polynomials and the Hessenberg matrix to write operator-theoretic results in terms of questions in rational approximation.

We have discussed here some asymptotics and location of the zeroes of the orthonormal polynomials. The same approach can be used to study less standard objects, like the quadrature formulas or the two-variable Padé approximants associated to a scalar product.

One can consider Sobolev products involving derivatives up to the k-th order. The Sobolev product takes the form:

$$\langle p, q\rangle = \sum_{i=0}^{k} p^{(k)}\overline{q^{(k)}} d\mu_k$$

Many of the results proven here have a straightforward generalization to this case. For example, (15) and (17) become

$$\sum_{i=0}^{k} \binom{n}{i} \left(S^t\right)^i M S^{k-i} = 0,$$
$$\sum_{i=0}^{k} \binom{n}{i} D^i \left(\overline{D}^t\right)^{k-i} = 0.$$

However, other questions like the discussion of Sobolev spaces deserve a careful examination. The theorems of [10] on the convergence of the finite sections method are also true for block Hessenberg matrices. Thus, in principle, the approach used here can also be applied to matrix orthogonal polynomials.

The theorems of the last section suggest that, once condition (5) is assumed, the Sobolev scalar product in the real line behaves similarly to a product obtained from a positive linear functional. Two directions of work can be followed. One is to refine the results proven here under (5). In particular, it is desirable to have a description of the set $Z^\infty(D)$. For instance, under what conditions $Z^\infty(D) \subset \mathbb{R}$? On the other hand, we can drop (5). A result in this direction is Theorem 10, proven in [6].

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