MATH 270–007 — FALL 2025

FINAL EXAM

SOLUTIONS

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1. Evaluate each of the following limits. You may use limit laws, algebraic manipulations and L'Hôpital's Rule (if applicable). If you use L'Hôpital's Rule, mark explicitly the step in which you use it. If the limit exists, give its value. If the limit equals ∞ or $-\infty$, then say so. If the limit does not exist and does **not** equal ∞ or $-\infty$, you must write explicitly DOES NOT EXIST. *(3 points each, 9 points total)*

   (a) $\displaystyle\lim_{x\to -1}\frac{2x^2-x+1}{x-1}$

   Answer. This is a rational function that is defined at -1, so we can simply evaluate:

$$\lim_{x\to -1}\frac{2x^2-x+1}{x-1} = \frac{2(-1)^2-(-1)+1}{(-1)-1} = \frac{2+1+1}{-1-1} = -\frac{4}{2} = -2.$$

   (b) $\displaystyle\lim_{x\to\infty}\frac{3x^2+1}{2x^2-x+1}$

   Answer. We can divide both numerator and denominator by the largest power of x in the denominator (namely x^2):

$$\lim_{x\to\infty}\frac{3x^2+1}{2x^2-x+1} = \lim_{x\to\infty}\frac{\frac{1}{x^2}(3x^2+1)}{\frac{1}{x^2}(2x^2-x+1)} = \lim_{x\to\infty}\frac{3+\frac{1}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}} = \frac{3+0}{2-0+0} = \frac{3}{2}.$$

   Alternatively, this is an $\frac{\infty}{\infty}$ indeterminate, so we can use L'Hopital's Rule twice:

$$\lim_{x\to\infty}\frac{3x^2+1}{2x^2-x+1} \stackrel{\text{LH}}{=} \lim_{x\to\infty}\frac{(3x^2+1)'}{(2x^2-x+1)'} = \lim_{x\to\infty}\frac{6x}{4x-1} \stackrel{\text{LH}}{=} \lim_{x\to\infty}\frac{(6x)'}{(4x-1)'} = \lim_{x\to\infty}\frac{6}{4} = \frac{6}{4} = \frac{3}{2}.$$

   (c) $\displaystyle\lim_{x\to 0^+} x\ln(x)$

   Answer. This is a $0\times\infty$ indeterminate. We rewrite in order to apply L'Hôpital's Rule:

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{(\ln x)'}{(\frac{1}{x})'} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}\frac{-x^2}{x} = \lim_{x\to 0^+}(-x) = 0.$$

2. On this page there are three functions. For each function, find the derivative $f'(x)$. Your final answer should contain no indicated derivatives, no compound fractions (fractions of fractions), and no easy arithmetic or algebraic simplifications left undone. But you are not required to obtain the simplest possible expression. *(3 points each, 9 points total)*

   (a) $f(x) = \arcsin(x)$

   Answer. This is one of our basic formulas: $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

   (b) $f(x) = \dfrac{x}{1+x^2}$

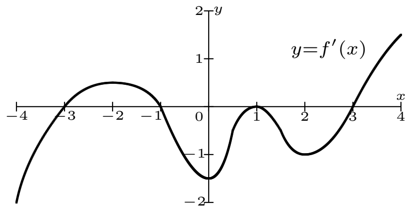
   Answer. A Quotient Rule:

$$f'(x) = \frac{(1+x^2)(x)'-x(1+x^2)'}{(1+x^2)^2} = \frac{(1+x^2)-x(2x)}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

(c) $f(x) = \tan(2 - 3x)$

Answer. A Chain Rule: $f'(x) = \left(\sec^2(2-3x)\right)(2-3x)' = -3\sec^2(2-3x)$.

3. The following sketch represents the graph of **the derivative** $f'(x)$ of a function that is defined and continuous everywhere. You may assume the function continues in the obvious manner before $x = -4$ and after $x = 4$. Use this graph to answer the following questions about the function $f(x)$. For parts (a) through (d), you do not need to justify your answer, you can simply give the answer. Part (e) requires justification. NOTE WELL: the questions are about the function $f(x)$, **NOT** about the function whose graph you are seeing.



Answer. I will provide an explanation for each, although you did not need one except in part (e).

(a) On what intervals is $f(x)$ increasing? *(2 points)*

Answer. This occurs on the intervals where $f'(x)$ is positive, including endpoints. Here, $[-3,-1]$ and $[3,\infty)$.

(b) On what intervals is $f(x)$ concave down? *(2 points)*

Answer. This occurs on the intervals where $f'(x)$ is decreasing, not including endpoints. Here, $(-2,0)$ and $(1,2)$.

(c) At what values of x does $f(x)$ have points of inflection? *(2 points)*

Answer. This occurs at the points where $f'(x)$ has relative extremes. Here, $x = -2$, $x = 0$, $x = 1$, and $x = 2$.

(d) What are the critical points of $f(x)$? *(2 points)*

Answer. This occurs at the points where $f'(x)$ is undefined, or $f'(x) = 0$. Here, $x = -3$, $x = -1$, $x = 1$, and $x = 3$.

(e) For each critical point, determine whether $f(x)$ has a local maximum, a local minimum, or neither at that point; justify your answer using either the First Derivative Test or the Second Derivative Test. *(4 points)*

Answer. Using the First Derivative Test: if $f'(x)$ changes from negative to positive at the critical point c, then f has a local minimum at c. This occurs at $x = -3$ and at $x = 3$.

If $f'(x)$ changes from positive to negative at the critical point c, then f has a local maximum at c. This happens at $x = -1$

If $f'(x)$ has the same sign on both sides of the critical point c, then $f(x)$ does not have a local extreme at c. This is what happens at $x = 1$.

Alternatively, using the Second Derivative Test: since $f'(x)$ is increasing while near -3 and near 3, then $f''(-3) > 0$ and $f''(3) > 0$, which tells us f has a local minimum at both $x = 3$ and $x = -3$. Since $f'(x)$ is decreasing near -1, $f''(-1) < 0$, so f has a local maximum at $x = -1$. But the test is inconclusive at $x = 1$, since there were have $f''(1) = 0$.

4. In this page and the next there are five integrals. Solve each integral by any valid method, **except plugging it into your calculator**. For indefinite integrals, give the most general antiderivative. For definite integrals, give the exact value. If you use a substitution, you must write down explicitly the substitution you are using, and the corresponding value of du. *(3 points each, 15 points total)*

(a) $\int \left(x^4 - 3x^3 + x^{-3} - 5\right)\, dx$

Answer. These are just Integration Power Rules, so

$$\int \left(x^4 - 3x^3 + x^{-3} - 5\right)\, dx = \frac{1}{5}x^5 - \frac{3}{4}x^4 - \frac{1}{2}x^{-2} - 5x + C.$$

(b) $\int_0^{\pi/12} 2\sec^2(3x)\, dx$

Answer. We do the substitution $u = 3x$; then $du = 3\, dx$, so $\frac{1}{3}\, du = dx$, and $\frac{2}{3}\, du = 2\, dx$. Remember to change the limits of integration: when $x = 0$, we have $u = 0$; when $x = \frac{\pi}{12}$, we have $u = 3(\frac{\pi}{12}) = \frac{\pi}{4}$. So

$$\begin{aligned}\int_0^{\pi/12} 2\sec^2(3x)\, dx &= \int_0^{\pi/4} \sec^2(u)\frac{2}{3}\, du = \frac{2}{3}\int_0^{\pi/4} \sec^2(u)\, du \\ &= \frac{2}{3}\tan(u)\Big|_0^{\pi/4} = \frac{2}{3}\tan\left(\frac{\pi}{4}\right) - \frac{2}{3}\tan(0) = \frac{2}{3}(1) - \frac{2}{3}(0) = \frac{2}{3}.\end{aligned}$$

(c) $\int_0^{\pi} \sin(x)\, dx$

Answer. We have

$$\int_0^{\pi} \sin(x)\, dx = -\cos(x)\Big|_0^{\pi} = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 1 + 1 = 2.$$

(d) $\int 2xe^{x^2}\, dx$

Answer. We substitute $u = x^2$, so $du = 2x\, dx$. Then we must return to x after computing the antiderivative:

$$\int 2xe^{x^2}\, dx = \int e^u\, du = e^u + C = e^{x^2} + C.$$

(e) $\int_{-2}^{2} x^3\, dx$

Answer. This integral equals 0 because $y = x^3$ is an odd function and we are integrating over the interval $[-2, 2]$. So $\int_{-2}^{2} x^3\, dx = 0$.

5. The following equation defines a function implicitly:

$$x^3 + y^3 = 4xy + 15.$$

Use implicit differentiation to find the equation of the tangent line to the graph at the point $(-1, 2)$. Express the equation of the tangent in the form $y = mx + b$.

Do not attempt to solve for y first. You must use implicit differentiation.

(8 points)

Answer. Note that the point $(-1, 2)$ is on the graph: plugging in $x = -1$ and $y = 2$ into the left hand side we have $(-1)^3 + (2)^3 = -1 + 8 = 7$, and the right to $4(-1)(2) + 15 = -8 + 15 = 7$. So $x = -1$, $y = 2$ satisfy the equation. We have:

$$\begin{aligned}x^3 + y^3 &= 4xy + 15 \\ \frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}(4xy + 15)\end{aligned}$$

$$
\begin{aligned}
3x^2 + 3y^2y' &= 4\big((x)'y + xy'\big) + 0 \\
3x^2 + 3y^2y' &= 4y + 4xy' \\
3y^2y' - 4xy' &= 4y - 3x^2 \\
(3y^2 - 4x)y' &= 4y - 3x^2 \\
y' &= \frac{4y - 3x^2}{3y^2 - 4x}.
\end{aligned}
$$

Plugging in, we have

$$
y'\Big|_{(-1,2)} = \frac{4(2) - 3(-1)^2}{3(2)^2 - 4(-1)} = \frac{8-3}{12+4} = \frac{5}{16}.
$$

So the slope of the tangent is $m = \frac{5}{16}$; the point is $(-1, 2)$. So the equation of the tangent line is

$$
\begin{aligned}
y - 2 &= \frac{5}{16}(x - (-1)) \\
y &= \frac{5}{16}x + \frac{5}{16} + 2 \\
y &= \frac{5}{16}x + \frac{37}{16}.
\end{aligned}
$$

6. At the annual sportsball game between the Springfield University Atoms and the Shelbyville Technical Institute Tireburners, Springfield's cheerleaders discover that the amount of time they spend singing their fight song, "Springfield Embiggens the Score," has a direct impact on the number of points their team scores. They have determined that if they sing for t minutes, then the Atoms will score

$$
A(t) = \frac{1}{2}t^2 - 4t + 50 \text{ points.}
$$

The cheerleaders can sing for any amount of time between 0 and 10 minutes. How long should they sing if they want to maximize the number of points Springfield scores? Show your work. *(8 points)*

Answer. We want to find the absolute maximum of the function $A(t)$ over the interval $[0, 10]$. This is a continuous function on a finite closed interval, so we just need to find the critical points, and evaluate at the critical points and the endpoints. The largest value will tell us how long the cheerleaders should sing. We have $A'(t) = t - 4$, which is defined everywhere, and equal to 0 at $t = 4$. So we need to evaluate A at 0, 4, and 10. We have:

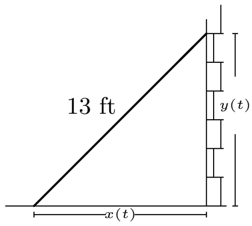
$$
\begin{aligned}
A(0) &= 50, \\
A(4) &= \frac{1}{2}(16) - 4(4) + 50 = 8 - 16 + 50 = 42, \\
A(10) &= \frac{1}{2}(100) - 4(10) + 50 = 50 - 40 + 50 = 60.
\end{aligned}
$$

So the maximum is achieved when $t = 10$.

That is, the cheerleaders should since for the full 10 minutes to maximize the number of points scored by Springfield.

7. A 13 foot ladder is leaning against a vertical wall. The foot of the ladder starts slipping away from the wall at a rate of $\frac{1}{2}$ ft/sec. The top of the ladder remains on the wall, but starts sliding down. How fast is the top of the ladder sliding down the instant that the foot of the ladder is 5 ft. from the wall? *(8 points)*

Answer. Let $x(t)$ be the distance from the foot of the ladder to the wall at time t, in feet, and $y(t)$ the distance from the top of the ladder to the ground at time t, also in feet. We measure t in seconds:



We are told that $\frac{dx}{dt} = 0.5$ ft/sec (positive because it is moving away from the wall). We want to know the value of $\frac{dy}{dt}$ when $x = 5$; it should be negative because $y(t)$ is getting smaller.

An equation that connects $x(t)$ and $y(t)$ is

$$13^2 = x^2 + y^2.$$

Differentiating implicitly, we get

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

We want to plug in the values of x and $\frac{dx}{dt}$ and solve for $\frac{dy}{dt}$; but we also need the value of y when $x = 5$.

If $13^2 = x^2 + y^2$, then at $x = 5$ we have $169 = 25 + y^2$, so $y^2 = 144$. Therefore, $y = 12$. Plugging in, we have

$$\begin{aligned} 0 &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ 0 &= 2(5)(0.5) + 2(12)\frac{dy}{dt} \\ -5 &= 24\frac{dy}{dt} \\ -\frac{5}{24} &= \frac{dy}{dt}. \end{aligned}$$

The result is negative, as we anticipated.

So when the foot of the ladder is 5 feet from the wall, the top of the ladder is sliding down the wall at a rate of $\frac{5}{24}$ feet per second.

8. Use the first and second derivatives of the function

$$f(x) = x^4 + 4x^3$$

to determine all intervals in which the function is both increasing and concave up (intervals in which both things happen). *(8 points)*

Answer. We have:

$$\begin{aligned} f(x) &= x^4 + 4x^3 \\ f'(x) &= 4x^3 + 12x^2 = 4x^2(x+3) \\ f''(x) &= 12x^2 + 24x = 12x(x+2). \end{aligned}$$

So the points where there can be a change of direction and/or concavity are $x = -3$, $x = -2$, and $x = 0$. We have:

	$(-\infty, -3)$	$(-3, -2)$	$(-2, 0)$	$(0, \infty)$
$f'(x)$	$-$	$+$	$+$	$+$
$f''(x)$	$+$	$+$	$-$	$+$
$f(x)$				

So the intervals where $f(x)$ is both increasing and concave up are $(-3, -2)$ and $(0, \infty)$.

9. We will use a linear approximation to approximate the value of

$$\sqrt{98}$$

by choosing an appropriate function $f(x)$, and a point a where $f(a)$ and $f'(a)$ are easy to compute.

(i) Write out the function $f(x)$, its derivative $f'(x)$, and the point a that you will use to approximate $\sqrt{98}$. *(3 points)*
Answer. While there are many possible answers, the simplest one is to take $f(x) = \sqrt{x}$, with $f'(x) = \frac{1}{2\sqrt{x}}$, and $a = 100$.

(ii) Give the exact values of $f(a)$ and $f'(a)$, written as whole numbers or irreducible fractions. *(1 point)*
Answer. The values are $f(100) = \sqrt{100} = 10$; and $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$.

(iii) Give the resulting approximation to the value of $\sqrt{98}$, using this linear approximation, written as an irreducible fraction. Do not approximate the approximation. *(2 points)*
Answer. The linear approximation says:

$$f(x) \approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a;$$
$$\sqrt{98} \approx 10 + \frac{1}{20}(98-100) = 10 - \frac{2}{20} = 10 - \frac{1}{10} = \frac{99}{10}.$$

So our approximation is that $\sqrt{98} \approx \frac{99}{10}$.

(iv) Use the second derivative of $f(x)$ to state whether your approximation is an overestimate or an underestimate. *(1 point)*
Answer. We have $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$. So $f''(100) < 0$. Since this means that the tangent line lies above the graph of the function, we conclude that our approximation was an OVERESTIMATE.

10. Determine the average value of the function $f(x) = \dfrac{1}{1+x^2}$ over the interval $[-1, 1]$. *(5 points)*

Answer. The average value is

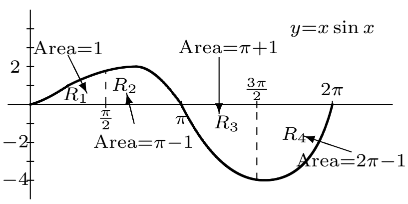
$$\overline{f} = \frac{1}{1-(-1)} \int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{1}{2}\arctan(x)\Big|_{-1}^{1} = \frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(-1) = \frac{\pi}{8} - \left(-\frac{\pi}{8}\right) = \frac{\pi}{4}.$$

11. Determine the net area between the the graph of $y = x^{-3} + 8$ and the x-axis, over the interval $[\frac{1}{2}, 1]$. Give the exact value, not an approximation. *(5 ponts)*

Answer. The net area is given by integral, so this is

$$\text{Net Area} = \int_{1/2}^{1} \left(x^{-3} + 8\right)\,dx = -\frac{1}{2}x^{-2} + 8x\,\Big|_{1/2}^{1} = \left(-\frac{1}{2} + 8\right) - (-2+4) = \frac{15}{2} - 2 = \frac{11}{2}.$$

12. The figure below shows four regions that are bounded by the graph of the function $y = x\sin x$, labeled R_1, R_2, R_3, and R_4; their areas are 1, $\pi - 1$, $\pi + 1$, and $2\pi - 1$, respectively. Those are the areas, **not** the net areas.



Use this information to evaluate the following integrals. Do not attempt to find an antiderivative for $f(x) = x \sin x$.

(1 point each, 6 points total)

(i) $\int_0^{\pi} x \sin x \, dx$

Answer. This the area of R_1 plus the area of R_2, so $\int_0^{\pi} x \sin x \, dx = 1 + (\pi - 1) = \pi$.

(ii) $\int_0^{3\pi/2} x \sin x \, dx$

Answer. This is the area of R_1, plus the area of R_2, *minus* the area of R_3, so

$$\int_0^{3\pi/2} x \sin x \, dx = 1 + (\pi - 1) - (\pi + 1) = -1.$$

(iii) $\int_0^{2\pi} x \sin x \, dx$

Answer. This is the area of R_1 plus the area of R_2, minus the area of R_3, minus the area of R_4. So

$$\int_0^{2\pi} x \sin x \, dx = 1 + (\pi - 1) - (\pi + 1) - (2\pi - 1) = -1 - 2\pi + 1 = -2\pi.$$

(iv) $\int_{3\pi/2}^{\pi/2} x \sin x \, dx$

Answer. Because we are integrating from right to left, this will be *minus* the area of R_2 plus the area of R_3

$$\int_{3\pi/2}^{\pi/2} x \sin x \, dx = -\int_{\pi/2}^{3\pi/2} x \sin x \, dx == -\Big((\pi - 1) - (\pi + 1)\Big) = -(-1 - 1) = 2.$$

(v) $\int_{\pi/2}^{2\pi} x \sin x \, dx$

Answer. This is the area of R_2, minus the area of R_3, minus the area of R_4. So we have

$$\int_{\pi/2}^{2\pi} x \sin x \, dx = (\pi - 1) - (\pi + 1) - (2\pi - 1) = -2 - 2\pi + 1 = -2\pi - 1.$$

(vi) $\int_{\pi}^{\pi} x \sin x \, dx$

Answer. This integral equals 0, because the lower limit of integration and the upper limit of integration are equal.