Math 270–005: Calculus I
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Homework 2
SOLUTIONS

§2.4

7. See the book for the graph.
   (a) As we approach 1 from the left, the values of f grow without bound. So $\lim_{x\to 1^-} f(x) = \infty$.
   (b) The same thing happens as we approach from the right, so $\lim_{x\to 1^+} f(x) = \infty$.
   (c) Since both one sided limits do not exist because the function "goes to infinity", the same is true for the two-sided limit: $\lim_{x\to 1} f(x) = \infty$.
   (d) As we approach 2 from the left, the values of f grow without bound, so $\lim_{x\to 2^-} f(x) = \infty$.
   (e) If we approach from the right, however, the values grow large and negative. So $\lim_{x\to 2^+} f(x) = -\infty$.
   (f) The two-sided limit does not exist; it does not equal either ∞ nor $-\infty$.
15. Factoring the numerator and denominator, we have

$$f(x) = \frac{x^3 - 4x + 3}{x^2 - 3x + 2} = \frac{(x-3)(x-1)}{(x-2)(x-1)}.$$

So the function is undefined at $x = 1$ and at $x = 2$.

As we approach 1, we have

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1}\frac{(x-3)(x-1)}{(x-2)(x-1)}\\ &= \lim_{x\to 1}\frac{x-3}{x-2} = \frac{1-3}{1-2}\\ &= \frac{-2}{-1} = 2.\end{aligned}$$

Since the limit exists, there is no vertical asymptote at $x = 1$.

When we approach 2, however, we have:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x\to 2^-}\frac{x-3}{x-2}.$$

If x is very small but a little less than 2, then $x-3$ will be very close to -1 and $x-2$ will be very small and negative. The quotient will be very large and positive, and grow larger the closer x is to 2. Therefore,

$$\lim_{x\to 2^-} f(x) = \infty.$$

This already tells us the function has a vertical asymptote at $x = 2$.

Something similar happens as we approach from the right:

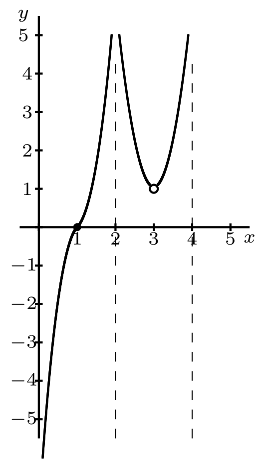
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\frac{x-3}{x-2}.$$

But this time, the denominator will be small and positive, so the quotient will be large and *negative* (since the numerator will be negative). So this time we have

$$\lim_{x\to 2^+} f(x) = -\infty.$$

This would also tell us that $f(x)$ has a vertical asymptote at $x = 2$.

17. There are of course many possibilities. Here is the sketch of one function defined on $[0, 4]$, with $f(1) = 0$, $f(3)$ undefined, $\lim_{x\to 3} f(x) = 1$, $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to 2} f(x) = \infty$, and $\lim_{x\to 4^-} f(x) = \infty$.



21. (a) As we approach 2 from the right, the denominator $x - 2$ is very small and positive, so the reciprocal is very large and positive. Therefore

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty.$$

(b) As we approach from the left, the main difference is that the denominator is now negative, so the reciprocal is large and negative. We then have

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty.$$

(c) Putting (a) and (b) together we conclude that the best we can say is that $\lim_{x\to 2} \frac{1}{x-2}$ does not exist.

23. (a) As x approaches 4 from the right, the denominator is small and positive, and the numerator approaches -1; the result is a large negative number. So

$$\lim_{x\to 4^+} \frac{x-5}{(x-4)^2} = -\infty.$$

(b) The same thing happens as we approach 4 from the left, because of the square in the denominator:

$$\lim_{x\to 4^-} \frac{x-5}{(x-4)^2} = -\infty.$$

(c) Since the one-sided limits are both ∞, we have

$$\lim_{x\to 4} \frac{x-5}{(x-4)^2} = -\infty.$$

25. (a) As we approach 3 from the right, the denominator is small and positive, and the numerator approaches 2, so the quotient is large and positive:

$$\lim_{z \to 3^+} \frac{(z-1)(z-2)}{(z-3)} = \infty.$$

(b) If we approach 3 from the left, however, the denominator is small and negative; the quotient will be large and negative. We therefore have

$$\lim_{z \to 3^-} \frac{(z-1)(z-2)}{(z-3)} = -\infty.$$

(c) The best we can say, then is that

$$\lim_{z \to 3} \frac{(z-1)(z-2)}{(z-3)} \text{ does not exist.}$$

27. (a) Note that $x^2 - 4x + 3$ takes value -1 at $x = 2$. So the denominator is very small and positive, the nuemrator is very close to -1, so

$$\lim_{x \to 2^+} \frac{x^2 - 4x + 3}{(x-2)^2} = -\infty.$$

(b) A similar thing happens here, because even though $x - 2$ is small and negative as we approach 2 from the left, once we square we get a small and positive number. We have:

$$\lim_{x \to 2^-} \frac{x^2 - 4x + 3}{(x-2)^2} = -\infty.$$

(c) Therefore we have $\lim_{x \to 2} \frac{x^2 - 4x + 3}{(x-2)^2} = -\infty$.

§2.5

17. While $\cos(\theta)$ oscillates between -1 and 1, when we divide by θ^2 we end up squeezing the cosine down to 0. We get

$$\lim_{\theta \to \infty} \frac{\cos \theta}{\theta^2} = 0.$$

21. The leading term goes to ∞, so $\lim_{x \to \infty} (3x^{12} - 9x^7) = \infty$.

25. Dividing both numerator and denominator by x^3 (the largest power of x that appears in the denominator) we have:

$$\begin{aligned}
\lim_{x \to \infty} \frac{14x^3 + 3x^2 - 2x}{21x^3 + x^2 + 2x + 1} &= \lim_{x \to \infty} \frac{\frac{1}{x^3}(14x^3 + 3x^2 - 2x)}{\frac{1}{x^3}(21x^3 + x^2 + 2x + 1)} \\
&= \lim_{x \to \infty} \frac{14 + \frac{3}{x} - \frac{2}{x^2}}{21 + \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}} \\
&= \frac{14 + 0 - 0}{21 + 0 + 0 + 0} = \frac{14}{21} = \frac{2}{3}.
\end{aligned}$$

37. We have that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x}{20x+1} = \lim_{x\to\infty} \frac{\frac{1}{x}(4x)}{\frac{1}{x}(20x+1)} = \lim_{x\to\infty} \frac{4}{20+\frac{1}{x}} = \frac{4}{20} = \frac{1}{5}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x}{20x+1} = \lim_{x\to-\infty} \frac{\frac{1}{x}(4x)}{\frac{1}{x}(20x+1)} = \lim_{x\to-\infty} \frac{4}{20+\frac{1}{x}} = \frac{4}{20} = \frac{1}{5}.$$

So the horizontal asymptote of $f(x)$ is $y = \frac{1}{5}$, both as $x \to \infty$ and as $x \to -\infty$.

41. Here we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^3-7}{x^4+5x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^4}(3x^3-7)}{\frac{1}{x^4}(x^4+5x^2)}$$
$$= \lim_{x\to\infty} \frac{\frac{3}{x}-\frac{7}{x^4}}{1+\frac{5}{x^2}} = \frac{0-0}{1+0} = 0.$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to infty} \frac{3x^3-7}{x^4+5x^2} = \lim_{x\to-\infty} \frac{\frac{1}{x^4}(3x^3-7)}{\frac{1}{x^4}(x^4+5x^2)}$$
$$= \lim_{x\to-\infty} \frac{\frac{3}{x}-\frac{7}{x^4}}{1+\frac{5}{x^2}} = \frac{0-0}{1+0} = 0.$$

So the line $y = 0$ is an asymptote to f for both the positive and negative directions.

73. (a) For horizontal asymptotes, we have:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to\infty} \frac{\frac{1}{x^4}(3x^4+3x^3-36x^2)}{\frac{1}{x^4}(x^4-25x^2+144)}$$
$$= \lim_{x\to\infty} \frac{3+\frac{3}{x}-\frac{36}{x^2}}{1-\frac{25}{x^2}+\frac{144}{x^4}} = \frac{3+0-0}{1-0+0} = 3.$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to-\infty} \frac{\frac{1}{x^4}(3x^4+3x^3-36x^2)}{\frac{1}{x^4}(x^4-25x^2+144)}$$
$$= \lim_{x\to-\infty} \frac{3+\frac{3}{x}-\frac{36}{x^2}}{1-\frac{25}{x^2}+\frac{144}{x^4}} = \frac{3+0-0}{1-0+0} = 3.$$

So the line $y = 3$ is a horizontal asyptote for $f(x)$ in both the positive and negative directions.

(b) We need to figure out the places where the denominator is 0, to find the possible places where vertical asymptotes may lie. Although the denominator is a degree 4 polynomial, it is actually a quadratic in x^2: letting $z = x^2$, we have

$$x^4-25x^2+144 = z^2-25z+144 = (z-9)(z-16) = (x^2-9)(x^2-16),$$

where the roots can be found using the quadratic formula. And now we just complete the factorization:

$$x^4-25x^2+144 = (x^2-9)(x^2-16) = (x-3)(x+3)(x-4)(x+4).$$

The possible locations of vertical asymptotes are thus $x = 3$, $x = -3$, $x = 4$, and $x = -4$. We need to find the limits to determine whether they are or not.
The numerator factors as

$$3x^4+3x^3-36x^2 = 3x^2(x^2+x-12) = 3x^2(x+4)(x-3).$$

We have:

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{3x^2(x+4)(x-3)}{(x-3)(x+3)(x-4)(x+4)} = \lim_{x\to 3} \frac{3x^2}{(x+3)(x-4)} = \frac{27}{(6)(-1)} = -\frac{27}{6}.$$

$$\lim_{x\to -4} f(x) = \lim_{x\to 3} \frac{3x^2}{(x+3)(x-4)} = \frac{3(16)}{(-1)(-8)} = 6.$$

So neither $x = 3$ nor $x = -4$ are vertical asymptotes.

As we approach -3, note that the numerator is always positive. If we approach -3 from the right, then $(x+3)$ is positive and $(x-4)$ negative, so the denominator is negative; if we approach from the left, then both $(x+3)$ and $(x-4)$ are negative, so the denominator will be positive. Thus,

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{3x^2}{(x-3)(x+4)} = -\infty,$$

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{3x^2}{(x-3)(x+4)} = \infty.$$

So $x = -3$ is a vertical asymptote for $f(x)$. Similarly, we have

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{3x^2}{(x-3)(x+4)} = \infty,$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{3x^2}{(x-3)(x+4)} = -\infty,$$

by analyzing the sign of the denominator in each case. So $x = 4$ is also a vertical asymptote for $f(x)$.

§2.6

17. Since the function

$$f(x) = \frac{2x^2+3x+1}{x^2+5x}$$

is not defined at $x = -5$ (the denominator evaluates to 0), the function cannote be continuous.

21. The function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1, \\ 3 & \text{if } x = 1, \end{cases}$$

is defined at $a = 1$. The value of the function is $f(1) = 3$. Now, we check the limit:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} (x+1) = 2.$$

Since $\lim_{x\to 1} f(x) \neq f(1)$, the function is *not* continuous at $a = 1$.

27. $f(x)$ is a rational function, which is undefined at $x = 3$ and at $x = -3$ (where the denominator $x^2 - 9$ is zero). Rational functions are continuous at all points where they are defined, so $f(x)$ is continuous on $(-\infty, -3)$, on $(-3, 3)$, and on $(3, \infty)$.

31. Because the "outside function" y^{40} is continuous everywhere, we can push the limit "in". We have:

$$\lim_{x\to 0} (x^8 - 3x^6 - 1)^{40} = \left(\lim_{x\to 0} (x^8 - 3x^6 - 1)\right)^{40} = (-1)^{40} = 1.$$

Alternatively, because $f(x) = (x^8 - 3x^6 - 1)^{40}$ is a polynomial, we have

$$\lim_{x\to 0} f(x) = f(0) = (-1)^{40} = 1.$$

39. We have

$$f(x) = \begin{cases} 2x & \text{if } x < 1, \\ x^2 + 3x & \text{if } x \geq 1. \end{cases}$$

We want to verify the function is not continuous at $a = 1$, and determine one sided continuity, if any.

(a) Note that $f(1) = 1 + 3 = 4$. To find the limit, the easiest thing to do is use the one-sided limits:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^3 + 3x) = 4,$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2.$$

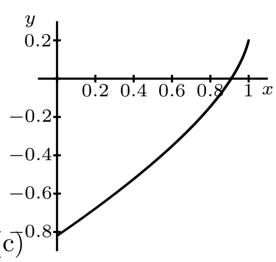
Since the one-sided limits are different, $\lim_{x \to 1} f(x)$ does not exist, so the function cannot be continuous at $a = 1$.

(b) We see above that the limit as $x \to 1^+$ is the same as $f(1)$, so the function is continuous from the right; but since $\lim_{x \to 1^-} f(x) = 2 \neq f(1)$, so the function is not continuous from the left.

(c) The function is continuous on $(-\infty, 1)$, because it is just a polynomial there. It is also continuous on $(1, \infty)$ because it is a polynomial there, and the continuity from the right at $a = 1$ lets us add that endpoint. So the intervals are $(-\infty, 1)$ and $[1, \infty)$.

67. (a) The function $f(x) = 2x^3 + x - 2$ is continuous everywhere, and so it is continuous on $[-1, 1]$. Since $f(-1) = -2 - 1 - 2 = -5$ and $f(1) = 2 + 1 - 2 = 1$, the function must pass through 0 somewhere in $(-1, 1)$, by the Intermediate Value Theorem.

(b) Graphing on the standard window suggests there is only one solution to $f(x) = 0$, and that it is near 1. Zooming in to look on the window $[0.7, 1] \times [-3, 1]$, suggests a value of about 0.835.



(c)