Math 270–005: Calculus I
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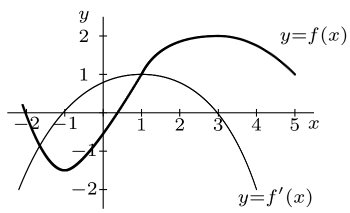
Homework 8
SOLUTIONS

§4.3

7. The thin graph is the given derivative. The other is one possible corresponding function.

   From the sign of the derivative we see that the function is decreasing up to -1; increasing on $[-1, 3]$, and decreasing after 3. Looking where the derivative is increasing and where it is increasing, we see the function is concave up up to -1, and concave down after.



23. We have
$$f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 4x$$
   and we want to determine the intervals on which f is increasing and the intervals on which it is decreasing.

   The function is defined everywhere. The derivative is:
$$f'(x) = x^2 - 5x + 4 = (x-4)(x-1).$$
   It is defined everywhere, and is zero at $x = 1$ and $x = 4$.

   On $(-\infty, 1)$, we have $f'(x) > 0$ since both factors are positive. On $(1, 4)$, we have $f'(x) < 0$; and on $(4, \infty)$, $f'(x)$ is positive.

   So $f(x)$ is increasing on $(-\infty, 1]$ and on $[4, \infty)$. It is decreasing on $[1, 4]$

29. We want to do the same now with $f(x) = x^2 \ln(x^2) + 1$. This is defined for $x \neq 0$.

   We have $f'(x) = 2x\ln(x^2) + x^2\left(\frac{2x}{x^2}\right) = 2x(\ln(x^2) + 1)$.

   This is zero only when $\ln(x^2) = -1$, which happens when $x^2 = e^{-1}$; this means $|x| = \frac{1}{\sqrt{e}}$.

   So we have three points where the derivative could change signs: at $x = -\frac{1}{\sqrt{e}}$, at $x = 0$, and at $x = \frac{1}{\sqrt{e}}$.

   On $(-\infty, -\frac{1}{\sqrt{e}})$, the derivative is negative; we can see this by plugging in -1, which lies in the interval, and we see we get $-2(\ln(1) + 1) = -2 < 0$. On $(-\frac{1}{\sqrt{e}}, 0)$, the derivative is positive; it is negative on $(0, \frac{1}{\sqrt{e}})$, and positive on $(\frac{1}{\sqrt{e}}, \infty)$.

   So $f(x)$ is increasing on $[-\frac{1}{\sqrt{e}}, 0)$ and on $[\frac{1}{\sqrt{e}}, \infty)$; and it is decreasing on $(-\infty, -\frac{1}{\sqrt{e}}]$ and on $(0, \frac{1}{\sqrt{e}})$.

36. We now look at $f(x) = x^2\sqrt{9-x^2}$ on $(-3,3)$.

We have

$$\begin{aligned} f'(x) &= 2x\sqrt{9-x^2} + x^2\left(\frac{-2x}{2\sqrt{9-x^2}}\right) \\ &= 2x\sqrt{9-x^2} - \frac{x^3}{\sqrt{9-x^2}} \\ &= \frac{2x(9-x^2)}{\sqrt{9-x^2}} - \frac{x^3}{\sqrt{9-x^2}} \\ &= \frac{18x - 2x^3 - x^3}{\sqrt{9-x^2}} \\ &= \frac{18x-3x^3}{\sqrt{9-x^2}} = \frac{3x(6-x^2)}{\sqrt{9-x^2}} \\ &= \frac{3x(\sqrt{6}-x)(\sqrt{6}+x)}{\sqrt{9-x^2}}. \end{aligned}$$

This always defined on $(-3,3)$; it is zero at $x = 0$, at $x = -\sqrt{6}$, and at $x = \sqrt{6}$.

If $-\sqrt{6} < x < \sqrt{6}$, then $6 - x^2 > 0$; and $6 - x^2 < 0$ otherwise. The denominator is always positive; and the sign of $3x$ depends on the sign of x. So we have:

On $(-3, -\sqrt{6})$, both factors iin the numerator are negative, so $f'(x) > 0$. On $(-\sqrt{6}, 0)$, we have $f'(x) < 0$. On $(0, \sqrt{6})$ we have $f'(x) > 0$; and on $(\sqrt{6}, 3)$ we have $f'(x) < 0$.

So $f(x)$ is increasing on $(-3, -\sqrt{6}]$ and on $[0, \sqrt{6}]$. And $f(x)$ is decreasing on $[-\sqrt{6}, 0]$ and on $[\sqrt{6}, 3)$.

47. The function is $f(x) = x\sqrt{4-x^2}$ on $[-2,2]$.

   (a) To find the critical points of $f(x)$, we compute the derivative:

$$\begin{aligned} f'(x) &= \sqrt{4-x^2} + x\left(\frac{-2x}{2\sqrt{4-x^2}}\right) \\ &= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \\ &= \frac{4-x^2}{\sqrt{4-x^2}} - \frac{x^2}{\sqrt{4-x^3}} \\ &= \frac{4-2x^2}{\sqrt{4-x^2}}. \end{aligned}$$

   This is defined everywhere in $(-2,2)$, but undefined at $x = -2$ and $x = 2$. These are the endpoints, so they are not relevant for local extremes.

   The other critical points occur when $f'(x) = 0$, which happens when $4 - 2x^2 = 0$. This means $x^2 = 2$. So we get $x = \sqrt{2}$ and $x = -\sqrt{2}$.

   So the critical points in $[-2,2]$ are $x = \sqrt{2}$ and $x = -\sqrt{2}$.

   (b) The denominator of the derivative is always positive. So the sign of the derivative only depends on the sign of $4 - 2x^2 = 2(2-x^2) = 2(\sqrt{2}-x)(\sqrt{2}+x)$.

   Before $-\sqrt{2}$, the derivative is negative; between $-\sqrt{2}$ and $\sqrt{2}$ we have $f'(x) > 0$. And for $x > \sqrt{2}$, the derivative is negative again.

By the First Derivative Test: at $x = -\sqrt{2}$, the derivative changes from negative to positive, so we have a local minimum. And at $x = \sqrt{2}$ the derivative changes from positive to negative, so the function has a local maximum. The local minimum is $f(-\sqrt{2}) = -\sqrt{2}\sqrt{2} = -2$. The local maximum is $f(\sqrt{2}) = \sqrt{2}\sqrt{2} = 2$.

(c) For the absolute extremes we also compare with the values at the endpoints. We have $f(2) = 2\sqrt{4-2^2} = 0$, and $f(-2) = -2\sqrt{4-(-2)^2} = 0$. So the absolute maximum of $f(x)$ on $[-2,2]$ is 2, achieved at $x = \sqrt{2}$, and the absolute minimum is -2, achieved at $x = -\sqrt{2}$.

77. Let $f(x) = x^3 - 3x^2$.

To find the critical points, we compute $f'(x) = 3x^2 - 6x = 3x(x-2)$. So the critical points are $x = 0$ and $x = 2$.

To apply the second derivative test, we calculate $f''(x) = 6x - 6$. We have $f''(0) = -6 < 0$ and $f''(2) = 6 > 0$.

Since $f'(0) = 0$ and $f''(0) < 0$, the Second Derivative Test tells us that f has a local maximum at $x = 0$. Because $f'(2) = 0$ and $f''(2) > 0$, it tells us that f has a local minimum at $x = 2$.

§4.4

19. We have the function $f(x) = x^4 - 6x^2$.

The function is defined everywhere. It is an even function: $f(-x) = (-x)^4 - 6(-x)^2 = x^4 - 6x^2 = f(x)$. So the graph is symmetric about the y-axis.

To find the shape of the graph, we take derivatives:

$$f'(x) = 4x^3 - 12x = 4x(x^2-3) = 4x(x+\sqrt{3})(x-\sqrt{3}),$$
$$f''(x) = 12x^2 - 12 = 12(x^2-1) = 12(x+1)(x-1).$$

The critical points are at $x = 0$, $x = -\sqrt{3}$ and at $x = \sqrt{3}$. The second derivative is 0 at $x = -1$ and at $x = 1$. Here is a sign chart:

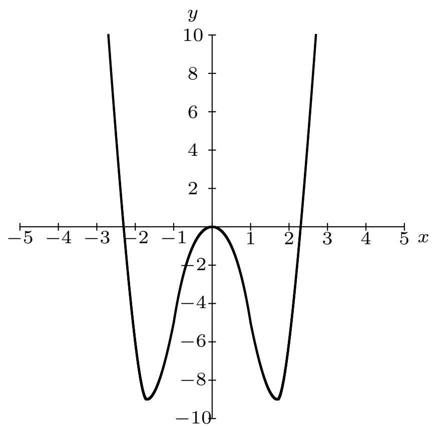
	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$-$	$+$	$+$	$-$	$-$	$+$
$f''(x)$	$+$	$+$	$-$	$-$	$+$	$+$
$f(x)$						

We have a local minimum at $x = \sqrt{3}$ and at $x = -\sqrt{3}$, with value $f(\sqrt{3}) = 9 - 18 = -9$. We have a local maximum at $x = 0$ with value $f(0) = 0$. We have a points of inflection at $x = 1$ and at $x = -1$.

The y-intercept is $f(0) = 0$. The x intercepts occur when $f(x) = x^4 - 6x^2 = 0$; This occurs when $x^2(x^2-6) = 0$, which is when $x = 0$ and when $x = -\sqrt{6}$ and $x = \sqrt{6}$.

There are no vertical asymptotes. And $\lim_{x\to\infty} f(x) = \infty$.

Putting it all together: on $[0, \infty)$, the function has interecepts at $(0,0)$ and at $(\sqrt{6}, 0)$. It has a local maximum of 0 at 0, and a local minimum of -9 at $x = \sqrt{3}$. IIt has a point of inflection at $x = 1$ (the point on the graph is $(1, -5)$, since $f(1) = -5$). And it goes to infinty as $x \to \infty$. The function is even. So we get the following picture:



28. For $f(x) = 3\sqrt{x} - x^{3/2} = \sqrt{x}(3-x)$.

The domain of the function is $[0,\infty)$, so there are no symmetries to find. It is not periodic.

We have:

$$f'(x) = \frac{3}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{3}{2}x^{-1/2}(1-x).$$
$$f''(x) = -\frac{3}{4}x^{-3/2} - \frac{3}{4}x^{-1/2} = -\frac{3}{4}x^{-3/2}(1+x).$$

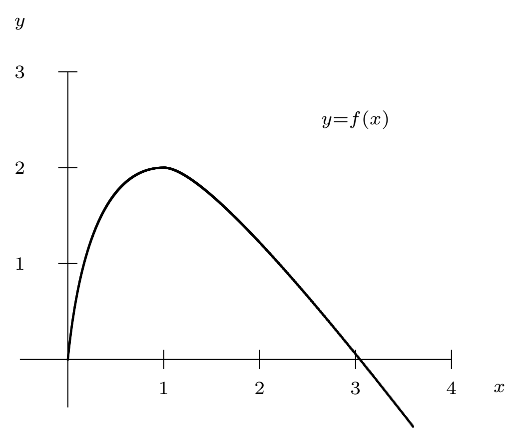
So the critical points on $[0,\infty)$ are $x=0$, which is an endpoint; and $x=1$. On the other hand, the second derivative is always negative, so $f(x)$ is concave down on $[0,\infty)$, with no points of inflection.

The derivative is positive on $(0,1)$ and negative on $(1,\infty)$; so $f(x)$ has a local maximum at $x=1$, with value $f(1)=2$.

The interecepts are $f(0)=0$, the origin. And the points where $f(x)=0$. Writing $f(x) = \sqrt{x}(3-x)$, we see that the other x-intercept occurs at $x=3$.

For the end behavior, we note that $\lim_{x\to\infty} f(x) = -\infty$, since the $-x^{3/2}$ term will dominate.

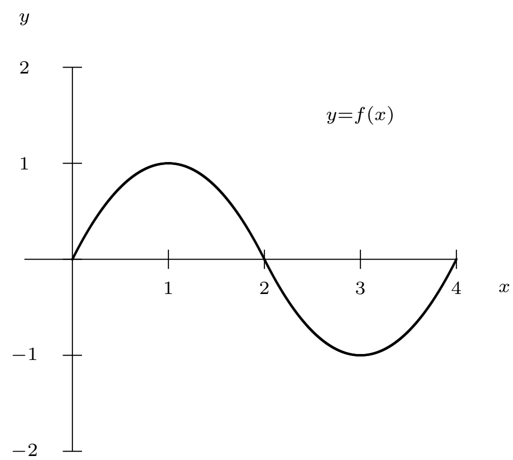
So we have the following sketch:



47. See the book for the graphs of $f'(x)$ and $f''(x)$ The function is increasing on $[0,1]$, decreasing on $[1,3]$, and increasing on $[3,4]$. It has a local maximum at $x=1$ and a local minimum at $x=3$.

The function is concave down on $(0, 2)$ and concave up on $(2, 4)$. It has a point of inflection at $x = 2$.

A possible graph with $f(0) = 0$ is:



57. Suppose the derivative of $f(x)$ is $f'(x) = 10\sin(2x)$, on $[-2\pi, 2\pi]$.

The derivative is 0 at $\pm 2\pi$, $\pm\frac{3\pi}{2}$, $\pm\pi$, $\pm\frac{\pi}{2}$, and at $x = 0$.

It is positive on $(-2\pi, -3\frac{\pi}{2})$, $(-\pi, -\frac{\pi}{2})$, $(0, \frac{\pi}{2})$, and $(\frac{3\pi}{2}, 2\pi)$. It is negative on the remaining portions.

There are local maxima at $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$. There are local minima at $x = -\pi$, $x = 0$, and $x = \pi$.

Here's an example of such a function: