Math 270–005: Calculus I
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Homework 9
Solutions

§4.5

19. If the side of the base has length x, and the box has height h, then the volume is $V = x^2h$, and the surface area is $S = 2x^2 + 4xh$ (the base and top have area x^2, the sides have area xh).

    Since we want the volume to be 8 m^3, we must have $x^2h = 8$, so $h = \frac{8}{x^2}$. Substituting into the surface, we get:
    $$S(x) = 2x^2 + 4x\left(\frac{8}{x^2}\right) = 2x^2 + \frac{32}{x}.$$
    The interval of interest is $(0,\infty)$, and we want to find the minimum.

    Note that $\lim\limits_{x\to\infty} S(x) = \infty$ and $\lim\limits_{x\to 0^+} S(x) = \infty$. So there should be a minimum.

    The critical points are given by $S'(x) = 4x - \frac{32}{x^2}$. This is zero when
    $$\begin{aligned} 4x - \frac{32}{x^2} &= 0\\ 4x &= \frac{32}{x^2}\\ x^3 &= 8\\ x &= 2. \end{aligned}$$
    Note that $S''(x) = 4 + \frac{64}{x^3}$, so $S''(2) > 0$; thus, the function has a local minimum at $x = 2$. Since this is the only relative extreme on the interval $(0,\infty)$, it is in fact an absolute minimum.

    The question asks for the dimensions of the box; we have concluded that the base is 2 meters by 2 meters. From our formula, we see that $h = \frac{8}{x^2} = \frac{8}{4} = 2$ meters.

    So the dimensions of the box of least surface area are $2 \times 2 \times 2$ meters: a cube.

21. Let the length of the side of the base be x, and the height h. The volume is fixed at $16 = x^2h$, so $h = \frac{16}{x^2}$.

    What is the total cost of a box with base length x and height h? The base costs twice as much per square foot as the sides; and the top costs half as much. If the cost per square foot of the sides is c (constant), then the total cost is: $(2c)x^2 + c(4xh) + \frac{c}{2}x^2$. Plugging in the value for h and simplifying, we get:
    $$\begin{aligned} f(x) &= 2cx^2 + 4cxh + \frac{c}{2}x^2\\ &= \left(2c + \frac{c}{2}\right)x^2 + 4cx\left(\frac{16}{x^2}\right)\\ &= \frac{5c}{2}x^2 + \frac{64c}{x}\\ &= c\left(\frac{5x^2}{2} + \frac{64}{x}\right). \end{aligned}$$
    We don't actually care about the value of c; it is positive, so it will simply scale the total. So we can set it to 1 and proceed.

We are working on the interval $(0,\infty)$. Note that $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(x) = \infty$, so there will be a minimum.

We have: $f'(x) = 5x - \frac{64}{x^2}$. This is zero when $x^3 = \frac{64}{5}$, which yields $x = \frac{4}{\sqrt[3]{5}}$.

Since $f''(x) = 5 + \frac{128}{x^3}$, the value at the critical point is positive. So $f(x)$ has a local minimum at $\frac{4}{\sqrt[3]{5}}$. Since this is the only local extreme on $(0,\infty)$, it is an absolute minimum.

Since $h = \frac{16}{x^2} = \frac{16\sqrt[3]{25}}{16} = \sqrt[3]{25}$, we conclude that the box-shaped crate of least cost has a $\frac{4}{\sqrt[3]{5}}$ by $\frac{4}{\sqrt[3]{5}}$ ft base, and a height of $\sqrt[3]{25}$ feet.

35. Let r and h be the radius and height of the can. Since the volume is 354 cubic centimeters, we have

$$V = 354 = \pi r^2 h, \qquad \text{or} \qquad h = \frac{354}{\pi r^2}.$$

The top and bottom of the can have area of πr^2; the side has area $(2\pi r)h$.

(a) If the top and bottom cost the same to make, then the total cost is proportional to the surface area. So we just need to figure out the surface area and minimize it. We have:

$$S = 2\pi r^2 + 2\pi r h = 2\pi\left(r^2 + r\left(\frac{354}{\pi r^2}\right)\right) = 2\pi\left(r^2 + \frac{354}{\pi r}\right).$$

We want to minimize over $(0,\infty)$.
The derivative is

$$S'(r) = 2\pi\left(2r - \frac{354}{\pi r^2}\right).$$

So the critical point has

$$\begin{aligned} 2r - \frac{354}{\pi r^2} &= 0 \\ 2r &= \frac{354}{\pi r^2} \\ r^3 &= \frac{354}{2\pi} \\ r &= \sqrt[3]{\frac{354}{2\pi}}. \end{aligned}$$

Note that $S''(r) = 2\pi\left(2 + \frac{690}{\pi r^3}\right)$.
Using the Second Derivative Test we see that it is a local minimum, and since it is the only local extreme on $(0,\infty)$, it is an absolute minimum. The corresponding value of h is

$$h = \frac{354}{\pi r^2} = \frac{354r}{\pi r^3} = \frac{354r}{\pi}\left(\frac{2\pi}{354}\right) = 2r.$$

So the diameter of the can equals its height. The can would look like a square seen from the side.

(b) If the top and bottom of the can have “double thickness”, then the cost function is not simply the sufrace area, but instead it is given by

$$F(r) = 2\pi\left(2r^2 + \frac{354}{\pi r}\right) = 4\pi\left(r^2 + \frac{177}{\pi r}\right).$$

Proceeding as above, we have

$$F'(r) = 4\pi \left(2r - \frac{177}{\pi r^2}\right),$$

so the only critical point lies at $r = \sqrt[3]{\frac{177}{2\pi}}$. It is an absolute minimum, and the corresponding height is

$$h = \frac{354r}{\pi r^3} = \frac{345r}{\pi}\left(\frac{2\pi}{177}\right) = 4r.$$

So now the height is four times the radius (twice the diameter), which matches with the dimensions given in the problem.

37. If we let x be the length of the flower garden and y the height, then we want $xy = 30$. Once we include the borders, the whole thing will have area

$$A = (x+4)(y+2) = xy + 4y + 2x + 8.$$

Substituting $y = \frac{30}{x}$, we obtain the function

$$A(x) = 30 + \frac{120}{x} + 2x + 8 = 2x + \frac{120}{x} + 38.$$

The domain is $(0, \infty)$.

The critical points are given by $A'(x) = 2 - \frac{120}{x^2}$. This gives $x^2 = 60$, so $x = \sqrt{60} = 2\sqrt{15}$.

The second derivative is $A''(x) = \frac{240}{x^3}$, so $A''(2\sqrt{15}) > 0$. Thus, we have a local minimum there. Since it is the only local extreme, it is an absolute minimum.

When $x = 2\sqrt{15}$, we have $y = \frac{30}{x} = \frac{2(15)}{2\sqrt{15}} = \sqrt{15}$.

Thus, the dimensons of the flower garden that minimze the combined area are $2\sqrt{15}$ meters by $\sqrt{15}$ meters, with the longer side having the 1 meter grass border.

46. Let x be the number of tickets sold; we will have $20 \le x \le 70$. The cost per person is $30 - \frac{1}{4}x$. The fixed expenses are 200. So we will receive $30 - \frac{1}{4}x$ dollars for each of the x tickets sold, and our total profit will be

$$P(x) = x\left(30 - \frac{1}{4}x\right) - 200 = 30x - \frac{1}{4}x^2 - 200.$$

The critical points are given by $P'((x) = 30 - \frac{x}{2}$, that is, $x = 60$.

Evaluating at $x = 20$, $x = 60$, and $x = 70$, we have:

$$\begin{aligned}
P(20) &= 30(20) - \frac{1}{4}(20)^2 - 200 = 600 - 100 - 200 = 300;\\
P(60) &= 30(60) - \frac{1}{4}(60)^2 - 200 = 1800 - 900 - 200 = 700;\\
P(70) &= 30(70) - \frac{1}{4}(70)^2 - 200 = 2100 - 1225 - 200 = 675.
\end{aligned}$$

So the maximum profit is achieved when we sell 60 tickets.

§4.6

37. We want to use a local linear approximation to approximate the value of $\frac{1}{203}$.

    There are many ways of setting this up. This is one of them: pick $f(x) = \frac{1}{x}$, and the point "nearby" where the function is easy to calculate is $x = 200$. We have $f(200) = \frac{1}{200} = 0.005$; and $f'(x) = -\frac{1}{x^2}$, so

$$f'(200) = -\frac{1}{40000} = -0.000025.$$

    The local linear approximation is:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a \\ f(x) &\approx \frac{1}{200} - \frac{1}{40000}(x-200) \qquad \text{for } x \text{ near } 200, \\ \frac{1}{203} = f(203) &\approx \frac{1}{200} - \frac{3}{40000} \\ &= \frac{200-3}{40000} = \frac{197}{40000} \\ &= 0.004925. \end{aligned}$$

    So $\frac{1}{203} \approx 0.004925$.

    (The actual value is approximately $0.0049261\ldots$).

41. Now we want to approximate $\ln(1.05)$. We can take $f(x) = \ln(x)$ (with $f'(x) = \frac{1}{x}$) and $a = 1$. We have $f(1) = 0$, and $f'(1) = \frac{1}{1} = 1$. So:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a \\ \ln(x) &\approx f(1) + f'(1)(x-1) \qquad \text{for } x \text{ near } 1 \\ \ln(1.05) &\approx 0 + (1.05-1) = 0.05. \end{aligned}$$

    (The actual value is $\ln(1.05) \approx 0.04879016\ldots$)

43. For $e^{0.06}$ we can take $f(x) = e^x$ (with $f'(x) = e^x$) and $a = 0$. We have $f(0) = f'(0) = 1$, so

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a \\ e^x &\approx f(0) + f'(0)(x-0) \qquad \text{for } x \text{ near } 0 \\ e^x &\approx 1 + x \qquad \text{for } x \text{ near } 0 \\ e^{0.06} &\approx 1 + 0.06 = 1.06. \end{aligned}$$

    This is close the actual value, which is $1.06183655\ldots$