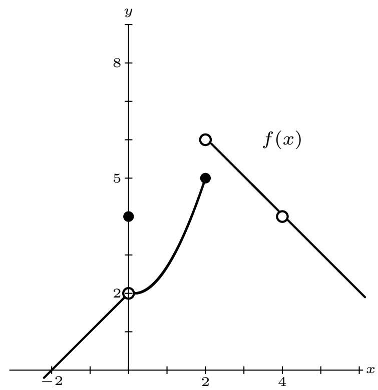
MATH 270-005 — SPRING 2024

CHAPTER 2 TEST

SOLUTIONS

Prof Arturo Magidin

1. Use the following figure to give values of the following limits or the function. If the limit does not exist, you must state explicitly it does not exist. If the function is not defined at the point, state explicitly that the value is undefined. You do not need to provide an explanation, just the answer. *(1 point each, 10 points total)*



Answers:

(i) $\lim\limits_{x\to 0^+} f(x) = 2$.

(ii) $\lim\limits_{x\to 0^-} f(x) = 2$.

(iii) $\lim\limits_{x\to 0} f(x) = 2$.

(iv) $f(0) = 4$.

(v) $\lim\limits_{x\to 2^+} f(x) = 6$.

(vi) $\lim\limits_{x\to 2^-} f(x) = 5$.

(vii) $\lim\limits_{x\to 2} f(x)$ does not exist.

(viii) $f(2) = 5$.

(ix) $\lim\limits_{x\to 4} f(x) = 4$.

(x) $f(4)$ is undefined.

2. In this page and the next are five limits. Use valid limit laws and algebraic manipulations to determine their value, or that they do not exist. You may **not** use sampling, or a graphing calculator, to try to compute the value of the limits. If the limit does not exist and is ∞ or $-\infty$, you must say so explicitly. If a limit does not exist but does not equal ∞ nor $-\infty$, then you must explicitly write that the limit DOES NOT EXIST. *(3 points each, 15 points total)*

(i) $\lim\limits_{x\to 1} \dfrac{5x^3 - 2x + 1}{x^2 - x + 1}$

Answer. The rational function is defined at $x = 1$, so we can just plug in:

$$\lim_{x\to 1} \frac{5x^3 - 2x + 1}{x^2 - x + 1} = \frac{5 - 2 + 1}{1 - 1 + 1} = \frac{4}{1} = 4.$$

(ii) $\lim_{x\to 2} \dfrac{x^2-5x+6}{x^2-3x+2}$

Answer. Trying to plug in $x=2$ gives 0 in both the numerator and the denominator. So we need to factor and cancel:

$$\lim_{x\to 2} \frac{x^2-5x+6}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x-3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x-3}{x-1} = \frac{2-3}{2-1} = \frac{-1}{1} = -1.$$

(iii) $\lim_{x\to \frac{\pi}{4}} \Big(\sin(x)+\cos(x)\Big)$

Answer. Both $\sin(x)$ and $\cos(x)$ are continuous everywhere so we can just plug in and evaluate:

$$\lim_{x\to \frac{\pi}{4}} \Big(\sin(x)+\cos(x)\Big) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}.$$

(iv) $\lim_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1}$

Answer. Trying to plug in results in zeros in both numerator and denominator. Multiply and divide by the conjugate to eliminate the radical, then simplify:

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} &= \lim_{x\to 1} \frac{\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)}{(x-1)\left(\sqrt{x+3}+2\right)} = \lim_{x\to 1} \frac{(x+3)-4}{(x-1)\left(\sqrt{x+3}+2\right)} \\ &= \lim_{x\to 1} \frac{x-1}{(x-1)\left(\sqrt{x+3}+2\right)} = \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.\end{aligned}$$

(v) $\lim_{x\to 0^+} \ln(x)$

Answer. One of our basic infinite limits, we have $\lim_{x\to 0^+} \ln(x) = -\infty$.

3. Let $f(x)$ be the function given by:

$$f(x) = \begin{cases} 2x & \text{if } x<1, \\ x^2+3x & \text{if } x\geq 1. \end{cases}$$

(i) Use the continuity checklist to show that f is not continuous at $a=1$. *(4 points)*

Answer. Is the function is defined at $a=1$? Yes, $f(1) = 1^2+3(1) = 4$.

Does the function have a limit at $a=1$? We check the one-sided limits:

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (x^2+3x) = 1+3 = 4, \\ \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 2x = 2.\end{aligned}$$

Since the one sided limits are different from each other, we have that

$$\lim_{x\to 1} f(x) \text{ does not exist.}$$

So $f(x)$ is not continuous at $a=1$.

(ii) Determine whether f is continuous from the right or from the left at $a=1$. *(4 points)*

Answer. From above, we have that $\lim_{x\to 1^+} f(x) = 4 = f(1)$, so f is continuous from the right at $a=1$.

Also, $\lim_{x\to 1^-} f(x) = 2 \neq f(1)$, so f is not continuous from the left at $a=1$.

(iii) State the intervals of continuity of $f(x)$. *(2 points)*

Answer. $f(x)$ is continuous on $(-\infty, 1)$, and on $[1, \infty)$. Note that we include 1 in the latter, but exclude it in the former.

4. Use the Intermediate Value Theorem to show that the equation

$$2x^3 + x - 2 = 0$$

has a solution on the interval $(-1, 1)$. *(5 points)*

Answer. Let $f(x) = 2x^3 + x - 2$. This is a polynomial, so it is continuous everywhere, and in particular it is continuous on $[-1, 1]$.

We want to find a point where $f(x) = 0$. We note that

$$\begin{gathered} f(-1) = -2 - 1 - 2 = -5 < 0, \\ f(1) = 2 + 1 - 2 = 1 > 0. \end{gathered}$$

So f has opposite signs on the endpoint of $[-1, 1]$. By the Intermediate Value Theorem, there exists at least one c, $-1 < c < 1$, where $f(c) = 0$. That is, a point c in $(-1, 1)$ where $2c^3 + c - 2 = 0$. This shows a solution to $2x^3 + x - 2 = 0$ exists on the interval $(-1, 1)$.

5. Let $f(x)$ be the following function:

$$f(x) = \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2} = \frac{2x(x+2)(x+3)}{x^2(x+2)}.$$

(i) State all the points at which $f(x)$ is not defined. *(1 point)*

Answer. This is a rational function. It is not defined precisely at the points where the denominator is equal to 0, which in this case happens at $x = 0$ and at $x = -2$.

(ii) Find $\lim\limits_{x\to\infty} f(x)$. *(2 points)*

Answer. We have:

$$\begin{aligned} \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2} &= \lim_{x\to\infty} \frac{\frac{1}{x^3}(2x^3 + 10x^2 + 12x)}{\frac{1}{x^3}(x^3 + 2x^2)} \\ = \lim_{x\to\infty} \frac{2 + \frac{10}{x} + \frac{12}{x^2}}{1 + \frac{2}{x}} &= \frac{2 + 0 + 0}{1 + 0} = 2. \end{aligned}$$

(iii) Use $\lim\limits_{x\to\infty} f(x)$ to determine if $y = f(x)$ has a horizontal asymptote as $x \to \infty$. If it does, give the equation of the horizontal asymptote. If it does not, state explicitly it does not have a horizontal asymptote as $x \to \infty$. *(1 point)*

Answer. From (ii), we conclude that $y = 2$ is the horizontal asymptote to $y = f(x)$ as $x \to \infty$.

(iv) Find $\lim\limits_{x\to-2} f(x)$, or show it does not exist. *(2 points)*

Answer. If we try plugging in we get zero divided by zero. Using the given factorization, we have:

$$\begin{aligned} \lim_{x\to-2} f(x) = \lim_{x\to-2} \frac{2x(x+2)(x+3)}{x^2(x+2)} &= \lim_{x\to-2} \frac{2x(x+3)}{x^2} \\ = \frac{2(-2)(-2+3)}{(-2)^2} &= \frac{4}{4} = 1. \end{aligned}$$

(v) Does $y = f(x)$ have a vertical asymptote at $x = -2$? Justify your answer. *(1 point)*

Answer. No. Since $\lim_{x \to -2} f(x)$ exists, neither of the one-sided limits can equal ∞ or $-\infty$, so there is no vertical asymptote at $x = -2$.

(vi) Find $\lim_{x \to 0^+} f(x)$ or show it does not exist. *(2 points)*

Answer. We have:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{2x(x+2)(x+3)}{x^2(x+2)} = \lim_{x \to 0^+} \frac{2(x+3)}{x} = \infty.$$

Note that for x small and positive, the numerator approaches 6, while the denominator is small and positive. So we get a large positive number from performing the operation. The quotient gets larger and larger as x gets closer and closer to 0, giving the limit as above.

(vii) Does $y = f(x)$ have a vertical asymptote at $x = 0$? Justify your answer. *(1 point)*

Answer. If at least one of the one-sided limits equals either ∞ or $-\infty$, then we have a vertical asymptote.

Since we have just verified that $\lim_{x \to 0^+} f(x) = \infty$, we conclude that yes, $y = f(x)$ has a vertical asymptote at $x = 0$.