Math 272: Honors Calculus I
Prof. Arturo Magidin
Homework 10
SOLUTIONS

§4.6

37. We want to use a local linear approximation to approximate the value of $\frac{1}{203}$.

There are many ways of setting this up. This is one of them: pick $f(x) = \frac{1}{x}$, and the point "nearby" where the function is easy to calculate is $x = 200$. We have $f(200) = \frac{1}{200} = 0.005$; and $f'(x) = -\frac{1}{x^2}$, so

$$f'(200) = -\frac{1}{40000} = -0.000025.$$

The local linear approximation is:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a \\ f(x) &\approx \frac{1}{200} - \frac{1}{40000}(x-200) \qquad \text{for } x \text{ near } 200, \\ \frac{1}{203} = f(203) &\approx \frac{1}{200} - \frac{3}{40000} \\ &= \frac{200-3}{40000} = \frac{197}{40000} \\ &= 0.004925. \end{aligned}$$

So $\frac{1}{203} \approx 0.004925$.

(The actual value is approximately $0.0049261\ldots$).

41. Now we want to approximate $\ln(1.05)$. We can take $f(x) = \ln(x)$ (with $f'(x) = \frac{1}{x}$) and $a = 1$. We have $f(1) = 0$, and $f'(1) = \frac{1}{1} = 1$. So:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a \\ \ln(x) &\approx f(1) + f'(1)(x-1) \qquad \text{for } x \text{ near } 1 \\ \ln(1.05) &\approx 0 + (1.05-1) = 0.05. \end{aligned}$$

(The actual value is $\ln(1.05) \approx 0.04879016\ldots$)

43. For $e^{0.06}$ we can take $f(x) = e^x$ (with $f'(x) = e^x$) and $a = 0$. We have $f(0) = f'(0) = 1$, so

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a \\ e^x &\approx f(0) + f'(0)(x-0) \qquad \text{for } x \text{ near } 0 \\ e^x &\approx 1 + x \qquad \text{for } x \text{ near } 0 \\ e^{0.06} &\approx 1 + 0.06 = 1.06. \end{aligned}$$

This is close the actual value, which is $1.06183655\ldots$

45. We want an approximation of $\frac{1}{\sqrt[3]{510}}$. We might realize that $512 = 2^9 = 8^3$ has an easy cubic root and is near 510. This suggests taking $a = 512$ and $f(x) = \frac{1}{x^{1/3}}$ (so that $f'(x) = -\frac{1}{3}x^{-4/3}$). We have $f(512) = \frac{1}{8}$, and $f'(512) = -\frac{1}{3}(8^{-4}) = -\frac{1}{3(4096)} = -\frac{1}{12288}$. So:

$$f(x) \approx f(a) + f'(a)(x-a) \qquad \text{for } x \text{ near } a$$

$$f(x) \approx f(512) + f'(512)(x - 512) \qquad \text{for } x \text{ near } 512$$
$$\frac{1}{\sqrt[3]{510}} = f(510) \approx \frac{1}{8} - \frac{1}{12288}(-2)$$
$$= \frac{1}{8} + \frac{1}{6144} = \frac{768+1}{6144} = \frac{769}{6144}.$$

This is approximately 0.12516276, which is very close to the value sought, which is approximately 0.125163186.

§4.7

24. This is an $\frac{\infty}{\infty}$ indeterminate. We can divide both numerator and denominator by x^3, or we can use L'Hopital's Rule. We have:

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{12x^2 - 4x}{3\pi x^2} = \lim_{x\to\infty} \left(\frac{4}{\pi} - \frac{4}{3\pi x}\right)$$
$$= \frac{4}{\pi} - 0 = \frac{4}{\pi}.$$

You could also apply L'Hopital's Rule twice momre to $\frac{12x^2-4x}{3\pi x^2}$ instead.

35. This is a $\frac{0}{0}$ indeterminate, and applying L'Hopital's Rule leads to another $\frac{0}{0}$ indeterminate, so we do it yet again. We have:

$$\lim_{x\to\pi} \frac{\cos x + 1}{(x-\pi)^2} \overset{\text{LH}}{=} \lim_{x\to\pi} \frac{-\sin x}{2(x-\pi)}$$
$$\overset{\text{LH}}{=} \lim_{x\to\pi} \frac{-\cos x}{2} = \frac{1}{2}.$$

53. This is an $\infty \times 0$ indeterminate. We rewrite and apply L'Hopital's Rule to the resulting $\frac{0}{0}$ indeterminate:

$$\lim_{x\to 0} x \csc x = \lim_{x\to 0} x \left(\frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{x}{\sin x}$$
$$\overset{\text{LH}}{=} \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1.$$

59. This is an $\infty - \infty$ indeterminate. We do the difference to turn it into a $\frac{0}{0}$ indeterminate. Then use L'Hopital's Rule again:

$$\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} \overset{\text{LH}}{=} \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\ln x + (x-1)\frac{1}{x}}$$
$$= \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\ln x + 1 - \frac{1}{x}} \overset{\text{LH}}{=} \lim_{x\to 1^+} \frac{-\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}}$$
$$= \lim_{x\to 1^+} \frac{-1}{x^2\left(\frac{1}{x} + \frac{1}{x^2}\right)} = \lim_{x\to 1^+} \frac{-1}{x+1}$$
$$= \frac{-1}{1+1} = -\frac{1}{2}.$$

§5.1

9. See the book for the graph. The length of each subinterval is $\Delta x = \frac{7-1}{6} = 1$. The values of the function at the grid points are

$$f(1) = 10,\ f(2) = 9,\ f(3) = 7,\ f(4) = 5,\ f(5) = 2,\ f(6) = 1\, f(7) = 0.$$

So the left Riemann sum is

$$f(1)(1) + f(2)(1) + f(3)(1) + f(4)(1) + f(5)(1) + f(6)(1) = 10 + 9 + 7 + 5 + 2 + 1 = 34.$$

The right Riemann sum is

$$f(2)(1) + f(3)(1) + f(4)(1) + f(5)(1) + f(6)(1) + f(7)(1) = 9 + 7 + 5 + 2 + 1 + 0 = 24.$$

15. See the book for the graphs:

(a) The function is $v = 3t^2 + 1$. The midpoints are $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$. So the approximation is:

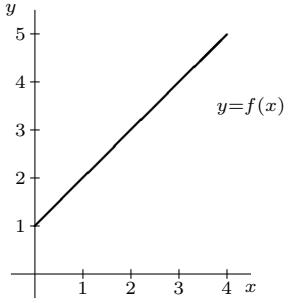
$$v(.5) + v(1.5) + v(2.5) + v(3.5) = 1.75 + 7.75 + 19.75 + 37.75 = 67 \text{ ft.}$$

(b) With the further divisions, now the interval length is $\frac{1}{2}$, and the midpoints are $\frac{1}{4}, \frac{3}{4}, \ldots, \frac{15}{4}$. So we have:

$$\begin{aligned} &v(0.25)(0.5) + v(0.75)(0.5) + v(1.25)(0.5) + v(1.75)(0.5) \\ &\qquad + v(2.25)(0.5) + v(2.75)(0.5) + v(3.25)(0.5) + v(3.75)(0.5) \\ &\qquad = 0.5(1.1875 + 2.6875 + 5.6875 + 10.1875 + 16.1875 + 23.6875 + 32.6875 + 43.1875) \\ &\qquad = 0.5(135.5) = 67.75 \text{ ft.} \end{aligned}$$

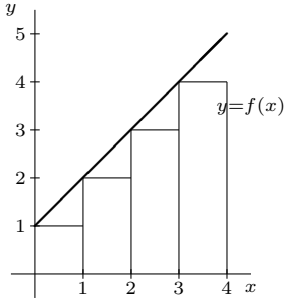
25. We have the function $f(x) = x + 1$ on the interval $[0, 4]$, and we want to divide the interval into $n = 4$ subintervals.

(a) First we want a sketch of the graph of the function, which is a straight line:



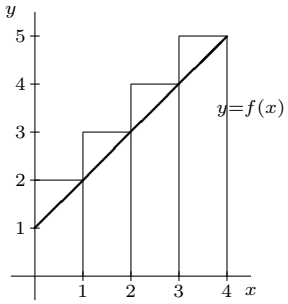
(b) Here $\Delta x = \frac{4-0}{4} = 1$, and the grid points are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$.

(c) The left Riemann sum is given by adding up the following rectangles:



From the picture, we see that this is an underestimate.

The right Riemann sum is given by adding the following rectangles:



From the picture we can see that this is an overestimate.

(d) The left Riemann sum is given by:

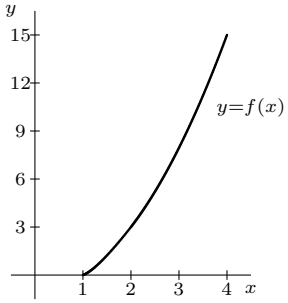
$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \sum_{k=1}^{4} f(x_{k-1})(1)$$
$$= f(0) + f(1) + f(2) + f(3) = 1 + 2 + 3 + 4 = 10.$$

The right Riemann sum is given by

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \sum_{k=1}^{4} f(x_k)(1)$$
$$= f(1) + f(2) + f(3) + f(4) = 2 + 3 + 4 + 5 = 14.$$

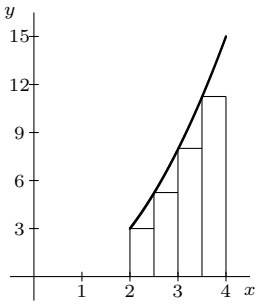
29. Now we want to do the same with $f(x) = x^2 - 1$ on $[2, 4]$, with $n = 4$.

(a) I use a compressed y-scale to save some space; the actual graph is three times taller; I also extend it to the left of 2 so that its curvature is a little more visible.



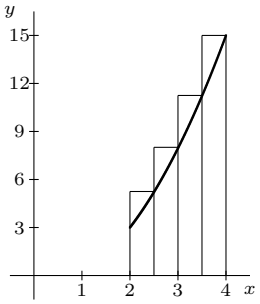
(b) Here we have $\Delta x = \frac{4-2}{4} = \frac{1}{2}$. The grid points are $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$, $x_3 = 3.5$, and $x_4 = 4$.

(c) The left Riemann sum is given by:



This is going to be an underestimate.

The right Riemann sum is given by:



This will yield an overestimate.

(d) The left Riemann sum is:

$$\begin{aligned}\sum_{k=1}^{4} f(x_k^*)\Delta x &= \sum_{k=1}^{4} f(x_{k-1})\left(\frac{1}{2}\right)\\ &= \frac{1}{2}\left(f(2)+f(2.5)+f(3)+f(3.5)\right)\\ &= \frac{1}{2}\left(3+\frac{21}{4}+8+\frac{45}{4}\right)\\ &= \frac{1}{2}\left(\frac{110}{4}\right)=\frac{55}{4}.\end{aligned}$$

The right Riemann sum is:

$$\begin{aligned}\sum_{k=1}^{4} f(x_k^*)\Delta x &= \sum_{k=1}^{4} f(x_k)\left(\frac{1}{2}\right)\\ &= \frac{1}{2}\left(f(2.5)+f(3)+f(3.5)+f(4)\right)\\ &= \frac{1}{2}\left(\frac{21}{4}+8+\frac{45}{4}+15\right)\\ &= \frac{1}{2}\left(\frac{158}{4}\right)=\frac{79}{4}.\end{aligned}$$