

**Math 272: Honors Calculus I**

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**Homework 2**

**SOLUTIONS**

**§2.3**

7. We are assuming that  $\lim_{x \rightarrow 1} f(x) = 8$ , so

$$\lim_{x \rightarrow 1} (4f(x)) = 4 \lim_{x \rightarrow 1} f(x) = 4(8) = 32.$$

11. Here, in addition to  $\lim_{x \rightarrow 1} f(x) = 8$ , we also assume  $\lim_{x \rightarrow 1} g(x) = 3$  and  $\lim_{x \rightarrow 1} h(x) = 2$ . We note that

$$\lim_{x \rightarrow 1} (g(x) - h(x)) = \lim_{x \rightarrow 1} g(x) - \lim_{x \rightarrow 1} h(x) = 3 - 2 = 1 \neq 0,$$

so we have

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x) - h(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} (g(x) - h(x))} = \frac{8}{1} = 8.$$

13. If  $\lim_{x \rightarrow 1} f(x) = 8$ , then  $\lim_{x \rightarrow 1} ((f(x))^{2/3}) = 8^{2/3} = 4$ .

15. If

$$g(x) = \begin{cases} 2x + 1 & \text{if } x \neq 0, \\ 5 & \text{if } x = 0, \end{cases}$$

then from the definition we see that  $g(0) = 5$ . And because  $g(x)$  takes the exact same values as  $2x + 1$  everywhere near 0, except *at* 0, we have

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (2x + 1) = 2(0) + 1 = 1.$$

19. To be explicit with the limit laws, we have

$$\lim_{x \rightarrow 4} (3x - 7) = 3 \lim_{x \rightarrow 4} x - \lim_{x \rightarrow 4} 7 = 3(4) - 7 = 5.$$

25. Since the limit of the denominator is not 0, we can proceed directly:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{5x^2 + 6x + 1}{8x - 4} &= \frac{\lim_{x \rightarrow 1} (5x^2 + 6x + 1)}{\lim_{x \rightarrow 1} (8x - 4)} \\ &= \frac{5(1)^2 + 6(1) + 1}{8(1) - 4} = \frac{5 + 6 + 1}{8 - 4} \\ &= \frac{12}{4} = 3. \end{aligned}$$

27. Since the limit of the denominator is not 0, we can proceed directly:

$$\begin{aligned} \lim_{p \rightarrow 2} \frac{3p}{\sqrt{4p+1}-1} &= \frac{\lim_{p \rightarrow 2} 3p}{\lim_{p \rightarrow 2} \sqrt{4p+1}-1} = \frac{3(2)}{\sqrt{4(2)+1}-1} \\ &= \frac{6}{\sqrt{9}-1} = \frac{6}{2} = 3. \end{aligned}$$

35. The denominator has a limit of 0; so we factor, and cancel:

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x^2 - 16}{4 - x} &= \lim_{x \rightarrow 4} \frac{(x - 4)(x + 4)}{-(x - 4)} \stackrel{*}{=} \lim_{x \rightarrow 4} \frac{x + 4}{-1} \\ &= -8.\end{aligned}$$

where  $*$  holds because the function  $\frac{(x-4)(x+4)}{4-x}$  and the function  $\frac{x+4}{-1}$  are equal at every point near  $x = 4$ , *except for* at  $x = 4$ ; so they have the same limit as  $x$  approaches 4.

36. If we try evaluating both numerator and denominator at  $t = 2$ , we get  $\frac{0}{0}$ . So we factor out  $t - 2$  from both, and cancel, as we did above:

$$\begin{aligned}\lim_{t \rightarrow 2} \frac{3t^2 - 7t + 2}{2 - t} &= \lim_{t \rightarrow 2} \frac{(t - 2)(3t - 1)}{-(t - 2)} \stackrel{*}{=} \lim_{t \rightarrow 2} \frac{3t - 1}{-1} \\ &= \lim_{t \rightarrow 2} -(3t - 1) = -(6 - 1) = -5,\end{aligned}$$

where  $*$  holds because the functions  $\frac{(t-2)(3t-1)}{(2-t)}$  and  $\frac{(3t-1)}{-1}$  are equal at all values of  $t$  *except for*  $t = 2$ ; so they have the same limit as  $t \rightarrow 2$ .

39. Again evaluating at  $x = -1$  gives 0 divided by 0; the potential problem is factoring  $x + 1$  out of the numerator. We can do that by first expanding the square:

$$(2x - 1)^2 - 9 = (4x^2 - 4x + 1) - 9 = 4x^2 - 4x - 8 = 4(x^2 - x - 2) = 4(x + 1)(x - 2).$$

So we have:

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{(2x - 1)^2 - 9}{x + 1} &= \lim_{x \rightarrow -1} \frac{4(x + 1)(x - 2)}{x + 1} \stackrel{*}{=} \lim_{x \rightarrow -1} \frac{4(x - 2)}{1} \\ &= 4(-1 - 2) = -12,\end{aligned}$$

where  $*$  holds because the two functions take the same values everywhere *except* at  $x = -1$ , so they have the same limit as  $x$  approaches  $-1$ .

49. This looks daunting, but if we do the algebra it becomes similar to problems we have already solved:

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{\frac{1}{x} - \frac{1}{4}}{x - 4} &= \lim_{x \rightarrow 4} \frac{\frac{4-x}{4x}}{x - 4} = \lim_{x \rightarrow 4} \frac{4 - x}{(4x)(x - 4)} \\ &= \lim_{x \rightarrow 4} \frac{-(x - 4)}{4x(x - 4)} = \lim_{x \rightarrow 4} \frac{-1}{4x} \\ &= -\frac{1}{16}.\end{aligned}$$

55. Multiply and divide by the conjugate of  $\sqrt{x} - 1$ ; this doesn't change the function; then we can cancel at  $*$  because the two functions agree everywhere except at  $x = 1$ :

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x} - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x} + 1)}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x} + 1)}{x - 1} \\ &\stackrel{*}{=} \lim_{x \rightarrow 1} \frac{\sqrt{x} + 1}{1} = \sqrt{1} + 1 = 1 + 1 = 2.\end{aligned}$$

73. The function is

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1, \\ \sqrt{x+1} & \text{if } x \geq -1. \end{cases}$$

- (a) Consider what happens as  $x$  approaches  $-1$  from the left. Since  $x$  will be smaller than  $-1$ , the function  $f$  just takes the value  $x^2 + 1$ . So we have

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (x^2 + 1) = (-1)^2 + 1 = 2.$$

- (b) On the other hand, if we approach  $-1$  from the right, the function  $f$  will take the exact same values as the function  $\sqrt{x+1}$ . Using the limit laws, we have

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \sqrt{x+1} = \sqrt{-1+1} = \sqrt{0} = 0.$$

- (c) Since the one-sided limits are not equal to each other,  $\lim_{x \rightarrow -1} f(x)$  does not exist.

87. We have the function

$$f(x) = \begin{cases} \frac{x^2 - 5x + 6}{x - 3} & \text{if } x \neq 3, \\ a & \text{if } x = 3. \end{cases}$$

We want to know what value of  $a$  will result in  $\lim_{x \rightarrow 3} f(x) = f(3)$ .

Note that  $f(3) = a$ ; so what we really need to find out is the value of  $\lim_{x \rightarrow 3} f(x)$ . Since  $f(x)$  takes the exact same values as  $\frac{x^2 - 5x + 6}{x - 3}$  for any  $x$  near (but not equal to)  $x$ , we have

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3}.$$

Now, this rational function has both a numerator and denominator that evaluate to 0 when we plug in 3, so we factor out  $x - 3$  from the numerator and cancel:

$$\begin{aligned} \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x - 3)(x - 2)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{x - 2}{1} = 3 - 2 = 1. \end{aligned}$$

So in order for  $f(3)$  to equal 1, we need  $a = 1$ .

89. We have

$$g(x) = \begin{cases} x^2 - 5x & \text{if } x \leq -1, \\ ax^3 - 7 & \text{if } x > -1. \end{cases}$$

We have

$$\lim_{x \rightarrow -1^-} g(x) = \lim_{x \rightarrow -1^-} (x^2 - 5x) = (-1)^2 - 5(-1) = 1 + 5 = 6.$$

From the right, we have

$$\lim_{x \rightarrow -1^+} g(x) = \lim_{x \rightarrow -1^+} (ax^3 - 7) = a(-1)^3 - 7 = -a - 7.$$

For the limit to exist, we need both of these one-sided limits to be equal. So we need

$$6 = -a - 7.$$

Solving for  $a$ , we have  $13 = -a$ , so  $a = -13$ . If that is the case, then we have

$$\lim_{x \rightarrow -1} g(x) = 6.$$

## §2.4

7. See the book for the graph.

- (a) As we approach 1 from the left, the values of  $f$  grow without bound. So  $\lim_{x \rightarrow 1^-} f(x) = \infty$ .
- (b) The same thing happens as we approach from the right, so  $\lim_{x \rightarrow 1^+} f(x) = \infty$ .
- (c) Since both one sided limits do not exist because the function “goes to infinity”, the same is true for the two-sided limit:  $\lim_{x \rightarrow 1} f(x) = \infty$ .
- (d) As we approach 2 from the left, the values of  $f$  grow without bound, so  $\lim_{x \rightarrow 2^-} f(x) = \infty$ .
- (e) If we approach from the right, however, the values grow large and negative. So  $\lim_{x \rightarrow 2^+} f(x) = -\infty$ .
- (f) The two-sided limit does not exist; it does not equal either  $\infty$  nor  $-\infty$ .

9. See the book for the graph.

- (a) As we approach  $-2$  from the left, the values of  $h(x)$  grow large and negative without bound. We have  $\lim_{x \rightarrow -2^-} h(x) = -\infty$ .
- (b) As we approach from the right, the same thing happens, so  $\lim_{x \rightarrow -2^+} h(x) = -\infty$ .
- (c) From (a) and (b) we conclude that  $\lim_{x \rightarrow -2} h(x) = -\infty$ .
- (d) As we approach 3 from the left, the values of  $h(x)$  grow large and positive. So  $\lim_{x \rightarrow 3^-} h(x) = \infty$ .
- (e) From the right, however, we have  $\lim_{x \rightarrow 3^+} h(x) = -\infty$ , from the graph.
- (f) The two-sided limit  $\lim_{x \rightarrow 3} h(x)$  does not exist. That is the best we can say.

15. Factoring the numerator and denominator, we have

$$f(x) = \frac{x^3 - 4x + 3}{x^2 - 3x + 2} = \frac{(x-3)(x-1)}{(x-2)(x-1)}.$$

So the function is undefined at  $x = 1$  and at  $x = 2$ .

As we approach 1, we have

$$\begin{aligned} \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{(x-3)(x-1)}{(x-2)(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{x-3}{x-2} = \frac{1-3}{1-2} \\ &= \frac{-2}{-1} = 2. \end{aligned}$$

Since the limit exists, there is no vertical asymptote at  $x = 1$ .

When we approach 2, however, we have:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{(x-3)(x-1)}{(x-2)(x-1)} = \lim_{x \rightarrow 2^-} \frac{x-3}{x-2}.$$

If  $x$  is very small but a little less than 2, then  $x-3$  will be very close to  $-1$  and  $x-2$  will be very small and negative. The quotient will be very large and positive, and grow larger the closer  $x$  is to 2. Therefore,

$$\lim_{x \rightarrow 2^-} f(x) = \infty.$$

This already tells us the function has a vertical asymptote at  $x = 2$ .

Something similar happens as we approach from the right:

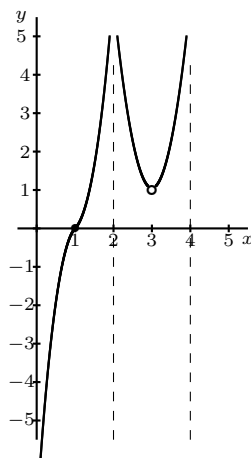
$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{x-3}{x-2}.$$

But this time, the denominator will be small and positive, so the quotient will be large and *negative* (since the numerator will be negative). So this time we have

$$\lim_{x \rightarrow 2^+} f(x) = -\infty.$$

This would also tell us that  $f(x)$  has a vertical asymptote at  $x = 2$ .

17. There are of course many possibilities. Here is the sketch of one function defined on  $[0, 4]$ , with  $f(1) = 0$ ,  $f(3)$  undefined,  $\lim_{x \rightarrow 3} f(x) = 1$ ,  $\lim_{x \rightarrow 0^+} f(x) = -\infty$ ,  $\lim_{x \rightarrow 2} f(x) = \infty$ , and  $\lim_{x \rightarrow 4^-} f(x) = \infty$ .



19. Both (a) and (b) are correct: we have

$$\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$$

which in particular means the limit does not exist. Note that if  $x$  is very near 1, then  $(x-1)$  is very close to 0, so  $(x-1)^2$  is very small and positive; the reciprocal will get larger and larger and remain positive.

Statement (c) is false because the values are always positive, the function cannot approach  $-\infty$ .

21. (a) As we approach 2 from the right, the denominator  $x - 2$  is very small and positive, so the reciprocal is very large and positive. Therefore

$$\lim_{x \rightarrow 2^+} \frac{1}{x - 2} = \infty.$$

- (b) As we approach from the left, the main difference is that the denominator is now negative, so the reciprocal is large and negative. We then have

$$\lim_{x \rightarrow 2^-} \frac{1}{x - 2} = -\infty.$$

- (c) Putting (a) and (b) together we conclude that the best we can say is that  $\lim_{x \rightarrow 2} \frac{1}{x - 2}$  does not exist.

23. (a) As  $x$  approaches 4 from the right, the denominator is small and positive, and the numerator approaches  $-1$ ; the result is a large negative number. So

$$\lim_{x \rightarrow 4^+} \frac{x - 5}{(x - 4)^2} = -\infty.$$

- (b) The same thing happens as we approach 4 from the left, because of the square in the denominator:

$$\lim_{x \rightarrow 4^-} \frac{x - 5}{(x - 4)^2} = -\infty.$$

- (c) Since the one-sided limits are both  $-\infty$ , we have

$$\lim_{x \rightarrow 4} \frac{x - 5}{(x - 4)^2} = -\infty.$$

25. (a) As we approach 3 from the right, the denominator is small and positive, and the numerator approaches 2, so the quotient is large and positive:

$$\lim_{z \rightarrow 3^+} \frac{(z - 1)(z - 2)}{(z - 3)} = \infty.$$

- (b) If we approach 3 from the left, however, the denominator is small and negative; the quotient will be large and negative. We therefore have

$$\lim_{z \rightarrow 3^-} \frac{(z - 1)(z - 2)}{(z - 3)} = -\infty.$$

- (c) The best we can say, then is that

$$\lim_{z \rightarrow 3} \frac{(z - 1)(z - 2)}{(z - 3)} \text{ does not exist.}$$

27. (a) Note that  $x^2 - 4x + 3$  takes value  $-1$  at  $x = 2$ . So the denominator is very small and positive, the numerator is very close to  $-1$ , so

$$\lim_{x \rightarrow 2^+} \frac{x^2 - 4x + 3}{(x - 2)^2} = -\infty.$$

- (b) A similar thing happens here, because even though  $x-2$  is small and negative as we approach 2 from the left, once we square we get a small and positive number. We have:

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 4x + 3}{(x-2)^2} = -\infty.$$

- (c) Therefore we have  $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 3}{(x-2)^2} = -\infty$ .

29. (a) If  $x$  is very close to 2 and slightly larger than 2, then  $x(x-2)$  is positive and very close to 0. Its square root is defined, and is very small and close to 0, so the reciprocal will get larger and larger. That is,

$$\lim_{x \rightarrow 2^+} \frac{1}{\sqrt{x(x-2)}} = \infty.$$

- (b) If  $x$  is very close to 2 and less than 2, then  $x-2 < 0$ , and  $x > 0$ , so  $x(x-2) < 0$ . Since it is negative, we cannot take the square root. That is,  $\sqrt{x(x-2)}$  is not defined to the immediate left of 2; that means that this limit does not exist, because we cannot even try to calculate it.
- (c) Since the limit from the left does not even make sense, let alone exist, this limit does not exist.