

Math 272: Honors Calculus I*Prof. Arturo Magidin***Homework 3****SOLUTIONS****§2.5**

17. While $\cos(\theta)$ oscillates between -1 and 1 , when we divide by θ^2 we end up squeezing the cosine down to 0 . We get

$$\lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta^2} = 0.$$

21. The leading term goes to ∞ , so $\lim_{x \rightarrow \infty} (3x^{12} - 9x^7) = \infty$.

25. Dividing both numerator and denominator by x^3 (the largest power of x that appears in the denominator) we have:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{14x^3 + 3x^2 - 2x}{21x^3 + x^2 + 2x + 1} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x^3}(14x^3 + 3x^2 - 2x)}{\frac{1}{x^3}(21x^3 + x^2 + 2x + 1)} \\ &= \lim_{x \rightarrow \infty} \frac{14 + \frac{3}{x} - \frac{2}{x^2}}{21 + \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}} \\ &= \frac{14 + 0 - 0}{21 + 0 + 0 + 0} = \frac{14}{21} = \frac{2}{3}. \end{aligned}$$

27. Dividing the numerator and denominator by x , we have:

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 3x}{x + 1} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x}(3x^2 + 3x)}{\frac{1}{x}(x + 1)} = \lim_{x \rightarrow -\infty} \frac{3x + 3}{1 + \frac{1}{x}} = -\infty.$$

33. We multiply and divide by the conjugate first; then we divide both the numerator and denominator by $x^2 = \sqrt{x^4}$:

$$\begin{aligned} \lim_{x \rightarrow \infty} (x^2 - \sqrt{x^4 + 3x^2}) &= \lim_{x \rightarrow \infty} \frac{(x^2 - \sqrt{x^4 + 3x^2})(x^2 + \sqrt{x^4 + 3x^2})}{x^2 + \sqrt{x^4 + 3x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{x^4 - (x^4 + 3x^2)}{x^2 + \sqrt{x^4 + 3x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{-3x^2}{x^2 + \sqrt{x^4 + 3x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}(-3x^2)}{\frac{1}{x^2}(x^2 + \sqrt{x^4 + 3x^2})} \\ &= \lim_{x \rightarrow \infty} \frac{-3}{1 + \frac{1}{\sqrt{x^4}}\sqrt{x^4 + 3x^2}} = \lim_{x \rightarrow \infty} \frac{-3}{1 + \sqrt{1 + \frac{3}{x^2}}} \\ &= \frac{-3}{1 + \sqrt{1 + 0}} = \frac{-3}{1 + 1} = -\frac{3}{2}. \end{aligned}$$

37. We have that

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{4x}{20x + 1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}(4x)}{\frac{1}{x}(20x + 1)} = \lim_{x \rightarrow \infty} \frac{4}{20 + \frac{1}{x}} = \frac{4}{20} = \frac{1}{5}. \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{4x}{20x + 1} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x}(4x)}{\frac{1}{x}(20x + 1)} = \lim_{x \rightarrow -\infty} \frac{4}{20 + \frac{1}{x}} = \frac{4}{20} = \frac{1}{5}. \end{aligned}$$

So the horizontal asymptote of $f(x)$ is $y = \frac{1}{5}$, both as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

41. Here we have

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{3x^3 - 7}{x^4 + 5x^2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^4}(3x^3 - 7)}{\frac{1}{x^4}(x^4 + 5x^2)} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{3}{x} - \frac{7}{x^4}}{1 + \frac{5}{x^2}} = \frac{0 - 0}{1 + 0} = 0. \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{3x^3 - 7}{x^4 + 5x^2} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^4}(3x^3 - 7)}{\frac{1}{x^4}(x^4 + 5x^2)} \\ &= \lim_{x \rightarrow -\infty} \frac{\frac{3}{x} - \frac{7}{x^4}}{1 + \frac{5}{x^2}} = \frac{0 - 0}{1 + 0} = 0.\end{aligned}$$

So the line $y = 0$ is an asymptote to f for both the positive and negative directions.

73. (a) For horizontal asymptotes, we have:

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^4}(3x^4 + 3x^3 - 36x^2)}{\frac{1}{x^4}(x^4 - 25x^2 + 144)} \\ &= \lim_{x \rightarrow \infty} \frac{3 + \frac{3}{x} - \frac{36}{x^2}}{1 - \frac{25}{x^2} + \frac{144}{x^4}} = \frac{3 + 0 - 0}{1 - 0 + 0} = 3. \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^4}(3x^4 + 3x^3 - 36x^2)}{\frac{1}{x^4}(x^4 - 25x^2 + 144)} \\ &= \lim_{x \rightarrow -\infty} \frac{3 + \frac{3}{x} - \frac{36}{x^2}}{1 - \frac{25}{x^2} + \frac{144}{x^4}} = \frac{3 + 0 - 0}{1 - 0 + 0} = 3.\end{aligned}$$

So the line $y = 3$ is a horizontal asymptote for $f(x)$ in both the positive and negative directions.

(b) We need to figure out the places where the denominator is 0, to find the possible places where vertical asymptotes may lie. Although the denominator is a degree 4 polynomial, it is actually a quadratic in x^2 : letting $z = x^2$, we have

$$x^4 - 25x^2 + 144 = z^2 - 25z + 144 = (z - 9)(z - 16) = (x^2 - 9)(x^2 - 16),$$

where the roots can be found using the quadratic formula. And now we just complete the factorization:

$$x^4 - 25x^2 + 144 = (x^2 - 9)(x^2 - 16) = (x - 3)(x + 3)(x - 4)(x + 4).$$

The possible locations of vertical asymptotes are thus $x = 3$, $x = -3$, $x = 4$, and $x = -4$. We need to find the limits to determine whether they are or not.

The numerator factors as

$$3x^4 + 3x^3 - 36x^2 = 3x^2(x^2 + x - 12) = 3x^2(x + 4)(x - 3).$$

We have:

$$\begin{aligned}\lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{3x^2(x + 4)(x - 3)}{(x - 3)(x + 3)(x - 4)(x + 4)} = \lim_{x \rightarrow 3} \frac{3x^2}{(x + 3)(x - 4)} = \frac{27}{(6)(-1)} = -\frac{27}{6}. \\ \lim_{x \rightarrow -4} f(x) &= \lim_{x \rightarrow -4} \frac{3x^2}{(x + 3)(x - 4)} = \frac{3(16)}{(-1)(-8)} = 6.\end{aligned}$$

So neither $x = 3$ nor $x = -4$ are vertical asymptotes.

As we approach -3 , note that the numerator is always positive. If we approach -3 from the right, then $(x + 3)$ is positive and $(x - 4)$ negative, so the denominator is negative; if we approach from the left, then both $(x + 3)$ and $(x - 4)$ are negative, so the denominator will be positive. Thus,

$$\begin{aligned}\lim_{x \rightarrow -3^+} f(x) &= \lim_{x \rightarrow -3^+} \frac{3x^2}{(x - 3)(x + 4)} = -\infty, \\ \lim_{x \rightarrow -3^-} f(x) &= \lim_{x \rightarrow -3^-} \frac{3x^2}{(x - 3)(x + 4)} = \infty.\end{aligned}$$

So $x = -3$ is a vertical asymptote for $f(x)$. Similarly, we have

$$\begin{aligned}\lim_{x \rightarrow 4^+} f(x) &= \lim_{x \rightarrow 4^+} \frac{3x^2}{(x - 3)(x + 4)} = \infty, \\ \lim_{x \rightarrow 4^-} f(x) &= \lim_{x \rightarrow 4^-} \frac{3x^2}{(x - 3)(x + 4)} = -\infty,\end{aligned}$$

by analyzing the sign of the denominator in each case. So $x = 4$ is also a vertical asymptote for $f(x)$.

75. (a) We have

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x^2 - 9}{x(x - 3)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}(x^2 - 9)}{\frac{1}{x^2}(x^2 - 3x)} \\ &= \lim_{x \rightarrow \infty} \frac{1 - \frac{9}{x^2}}{1 - \frac{3}{x}} = 1. \\ \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{x^2 - 9}{x(x - 3)} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}(x^2 - 9)}{\frac{1}{x^2}(x^2 - 3x)} \\ &= \lim_{x \rightarrow -\infty} \frac{1 - \frac{9}{x^2}}{1 - \frac{3}{x}} = 1.\end{aligned}$$

So $y = 1$ is a horizontal asymptote in both the positive and negative directions.

(b) The possible vertical asymptotes are at $x = 0$ and at $x = 3$. We have:

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2 - 9}{x(x - 3)} &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x(x - 3)} = \lim_{x \rightarrow 3} \frac{x + 3}{x} = 2. \\ \lim_{x \rightarrow 0^+} \frac{x^2 - 9}{x(x - 3)} &= \lim_{x \rightarrow 0^+} \frac{x + 3}{x} = \infty, \\ \lim_{x \rightarrow 0^-} \frac{x^2 - 9}{x(x - 3)} &= \lim_{x \rightarrow 0^-} \frac{x + 3}{x} = -\infty,\end{aligned}$$

so there is a vertical asymptote at $x = 0$, but there is no asymptote at $x = 3$.

§2.6

17. Since the function

$$f(x) = \frac{2x^2 + 3x + 1}{x^2 + 5x}$$

is not defined at $x = -5$ (the denominator evaluates to 0), the function cannot be continuous.

21. The function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1, \\ 3 & \text{if } x = 1, \end{cases}$$

is defined at $a = 1$. The value of the function is $f(1) = 3$. Now, we check the limit:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \rightarrow 1} (x+1) = 2.$$

Since $\lim_{x \rightarrow 1} f(x) \neq f(1)$, the function is *not* continuous at $a = 1$.

27. $f(x)$ is a rational function, which is undefined at $x = 3$ and at $x = -3$ (where the denominator $x^2 - 9$ is zero). Rational functions are continuous at all points where they are defined, so $f(x)$ is continuous on $(-\infty, -3)$, on $(-3, 3)$, and on $(3, \infty)$.

31. Because the “outside function” y^{40} is continuous everywhere, we can push the limit “in”. We have:

$$\lim_{x \rightarrow 0} (x^8 - 3x^6 - 1)^{40} = \left(\lim_{x \rightarrow 0} (x^8 - 3x^6 - 1) \right)^{40} = (-1)^{40} = 1.$$

Alternatively, because $f(x) = (x^8 - 3x^6 - 1)^{40}$ is a polynomial, we have

$$\lim_{x \rightarrow 0} f(x) = f(0) = (-1)^{40} = 1.$$

39. We have

$$f(x) = \begin{cases} 2x & \text{if } x < 1, \\ x^2 + 3x & \text{if } x \geq 1. \end{cases}$$

We want to verify the function is not continuous at $a = 1$, and determine one sided continuity, if any.

(a) Note that $f(1) = 1 + 3 = 4$. To find the limit, the easiest thing to do is use the one-sided limits:

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (x^2 + 3x) = 4, \\ \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} 2x = 2. \end{aligned}$$

Since the one-sided limits are different, $\lim_{x \rightarrow 1} f(x)$ does not exist, so the function cannot be continuous at $a = 1$.

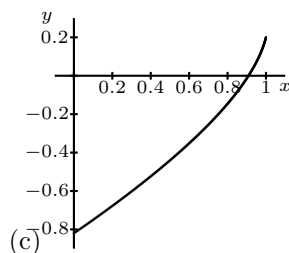
(b) We see above that the limit as $x \rightarrow 1^+$ is the same as $f(1)$, so the function is continuous from the right; but since $\lim_{x \rightarrow 1^-} f(x) = 2 \neq f(1)$, so the function is not continuous from the left.

(c) The function is continuous on $(-\infty, 1)$, because it is just a polynomial there. It is also continuous on $(1, \infty)$ because it is a polynomial there, and the continuity from the right at $a = 1$ lets us add that endpoint. So the intervals are $(-\infty, 1)$ and $[1, \infty)$.

41. The function $f(x) = \sqrt{5-x}$ is defined wherever $5-x \geq 0$, that is, on $(-\infty, 5]$. It is continuous from the left at 5, so the interval is $(-\infty, 5]$.

67. (a) The function $f(x) = 2x^3 + x - 2$ is continuous everywhere, and so it is continuous on $[-1, 1]$. Since $f(-1) = -2 - 1 - 2 = -5$ and $f(1) = 2 + 1 - 2 = 1$, the function must pass through 0 somewhere in $(-1, 1)$, by the Intermediate Value Theorem.

- (b) Graphing on the standard window suggests there is only one solution to $f(x) = 0$, and that it is near 1. Zooming in to look on the window $[0.7, 1] \times [-3, 1]$, suggests a value of about 0.835.



71. (a) The function $f(x) = x + e^x$ is continuous everywhere, so it is continuous on $[-1, 0]$. We have $f(-1) = -1 + \frac{1}{e} < 0$, and $f(0) = e^0 = 1$. So by the Intermediate Value Theorem, there is (at least) one point c , $-1 < c < 0$, where $f(c) = 0$.
- (b) Zooming in with a graphic calculator shows that the value seems to be somewhere around -0.55.

