

Math 272: Honors Calculus I*Prof. Arturo Magidin***Homework 4****SOLUTIONS****§3.1**

13. If $s(t) = -16t^2 + 100t$ is the position at time t , then the instantaneous velocity at time $a = 1$ is:

$$\begin{aligned} s'(1) &= \lim_{t \rightarrow 1} \frac{s(t) - s(1)}{t - 1} = \lim_{t \rightarrow 1} \frac{(-16t^2 + 100t) - 84}{t - 1} = \lim_{t \rightarrow 1} \frac{-16t^2 + 100t - 84}{t - 1} \\ &= \lim_{t \rightarrow 1} \frac{(t - 1)(-16t + 84)}{t - 1} = \lim_{t \rightarrow 1} \frac{-16t + 84}{1} = -16 + 84 = 68. \end{aligned}$$

So the velocity at $t = 1$ is 68 feet/sec.

15. The function is $f(x) = x^2 - 5$; the point is $P(3, 4)$ (note that, indeed, $f(3) = 4$).

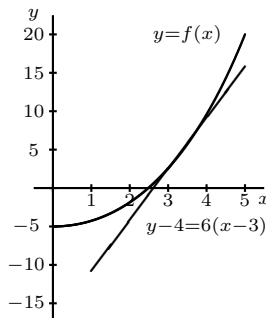
(a) Using Definition 1, we have that the slope is:

$$\begin{aligned} m_{\tan} &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{(x^2 - 5) - 4}{x - 3} = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3} \frac{x + 3}{1} = 3 + 3 = 6. \end{aligned}$$

(b) The equation of the tangent is

$$\begin{aligned} y - f(a) &= m_{\tan}(x - a) \\ y - 4 &= 6(x - 3). \end{aligned}$$

(c) Here is a plot of the region near the point $(3, 4)$:



21. The function is $f(x) = 2x + 1$, the point is $(0, 1)$: note that $f(0) = 1$.

(a) Here we have, using Definition (2),

$$\begin{aligned} m_{\tan} &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{(2h + 1) - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h} = \lim_{h \rightarrow 0} \frac{2}{1} = 2. \end{aligned}$$

(b) The equation of the tangent is given by

$$\begin{aligned}y - f(a) &= m_{\tan}(x - a) \\y - 1 &= 2(x - 0) \\y - 1 &= 2x.\end{aligned}$$

31. Here the function is $f(x) = \sqrt{x+3}$, the point is $(1, 2)$; again note that we do indeed have $f(1) = 2$.

(a) We have:

$$\begin{aligned}m_{\tan} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1+h+3} - \sqrt{1+3}}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{h+4} - 2}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{h+4} - 2)(\sqrt{h+4} + 2)}{h(\sqrt{h+4} + 2)} \\&= \lim_{h \rightarrow 0} \frac{(h+4) - 4}{h(\sqrt{h+4} + 2)} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{h+4} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2+2} = \frac{1}{4}.\end{aligned}$$

(b) The equation of the tangent is then given by

$$\begin{aligned}y - f(a) &= m_{\tan}(x - a) \\y - 2 &= \frac{1}{4}(x - 1).\end{aligned}$$

35. Here $f(x) = 4x^2 + 2x$, with $a = -2$.

(a) We want to find $f'(a)$. We have:

$$\begin{aligned}f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \rightarrow 0} \frac{4(h-2)^2 + 2(h-2) - (16-4)}{h} \\&= \lim_{h \rightarrow 0} \frac{4(h^2 - 4h + 4) + 2h - 4 - 12}{h} = \lim_{h \rightarrow 0} \frac{4h^2 - 16h + 16 + 2h - 4 - 12}{h} \\&= \lim_{h \rightarrow 0} \frac{4h^2 - 14h}{h} = \lim_{h \rightarrow 0} \frac{h(4h - 14)}{h} \\&= \lim_{h \rightarrow 0} \frac{4h - 14}{1} = -14.\end{aligned}$$

(b) The tangent goes through $(a, f(a)) = (-2, 12)$, with slope -14 . So the equation is:

$$y - 12 = -14(x + 2).$$

37. Here $f(x) = \frac{1}{\sqrt{x}}$, and $a = \frac{1}{4}$.

(a) Note that $\sqrt{\frac{1}{4}} = \frac{1}{2}$, so $f(\frac{1}{4}) = \frac{1}{\sqrt{\frac{1}{4}}} = 2$. We have

$$\begin{aligned}f'\left(\frac{1}{4}\right) &= \lim_{x \rightarrow \frac{1}{4}} \frac{f(x) - f(\frac{1}{4})}{x - \frac{1}{4}} = \lim_{x \rightarrow \frac{1}{4}} \frac{\frac{1}{\sqrt{x}} - 2}{x - \frac{1}{4}} = \lim_{x \rightarrow \frac{1}{4}} \frac{\frac{1-2\sqrt{x}}{\sqrt{x}}}{x - \frac{1}{4}} \\&= \lim_{x \rightarrow \frac{1}{4}} \frac{1-2\sqrt{x}}{\sqrt{x}(x - \frac{1}{4})} = \lim_{x \rightarrow \frac{1}{4}} \frac{(1-2\sqrt{x})(1+2\sqrt{x})}{\sqrt{x}(x - \frac{1}{4})(1+2\sqrt{x})}\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow \frac{1}{4}} \frac{(1-4x)}{\sqrt{x}(x-\frac{1}{4})(1+2\sqrt{x})} = \lim_{x \rightarrow \frac{1}{4}} \frac{4(\frac{1}{4}-x)}{\sqrt{x}(x-\frac{1}{4})(1+2\sqrt{x})} \\
&= \lim_{x \rightarrow \frac{1}{4}} \frac{-4}{\sqrt{x}(1+2\sqrt{x})} = \lim_{x \rightarrow \frac{1}{4}} \frac{-4}{\sqrt{x}(1+2\sqrt{x})} \\
&= \frac{-4}{\sqrt{\frac{1}{4}}(1+2\sqrt{\frac{1}{4}})} = \frac{-4}{\frac{1}{2}(1+2(\frac{1}{2}))} = \frac{-4}{\frac{1}{2}(1+1)} = -4.
\end{aligned}$$

(b) The tangent goes through $(\frac{1}{4}, 2)$, and has slope -4 , so the equation is:

$$y - 2 = -4 \left(x - \frac{1}{4} \right).$$

43. The derivative of $f(t) = \frac{1}{t+1}$ at $a = 1$ is

$$\begin{aligned}
f'(1) &= \lim_{t \rightarrow 1} \frac{f(t) - f(1)}{t - 1} = \lim_{t \rightarrow 1} \frac{\frac{1}{t+1} - \frac{1}{2}}{t - 1} = \lim_{t \rightarrow 1} \frac{\frac{2-(t+1)}{2(t+1)}}{t - 1} \\
&= \lim_{t \rightarrow 1} \frac{1-t}{2(t+1)(t-1)} = \lim_{t \rightarrow 1} \frac{-(t-1)}{2(t+1)(t-1)} = \lim_{t \rightarrow 1} \frac{-1}{2(t+1)} = -\frac{1}{4}.
\end{aligned}$$

§3.2

21. (a) The function $f(x) = 5x + 2$ has derivative:

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(5(x+h) + 2) - (5x + 2)}{h} = \lim_{h \rightarrow 0} \frac{5x + 5h + 2 - 5x - 2}{h} \\
&= \lim_{h \rightarrow 0} \frac{5h}{h} = \lim_{h \rightarrow 0} \frac{5}{1} = 5.
\end{aligned}$$

So $f'(x) = 5$.

(b) That means that $f'(1) = 5$ and $f'(2) = 5$.

23. (a) Here we have $f(x) = 4x^2 + 1$. We have:

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(4(x+h)^2 + 1) - (4x^2 + 1)}{h} \\
&= \lim_{h \rightarrow 0} \frac{4(x^2 + 2xh + h^2) + 1 - 4x^2 - 1}{h} = \lim_{h \rightarrow 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{8xh + 4h^2}{h} = \lim_{h \rightarrow 0} \frac{h(8x + 4h)}{h} \\
&= \lim_{h \rightarrow 0} \frac{8x + 4h}{1} = 8x.
\end{aligned}$$

So $f'(x) = 8x$.

(b) So we have $f'(2) = 16$ and $f'(4) = 32$.

25. (a) Now we have $f(x) = \frac{1}{x+1}$; so

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \lim_{h \rightarrow 0} \frac{\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{x+1-x-h-1}{(x+h+1)(x+1)h} = \lim_{h \rightarrow 0} \frac{-h}{(x+h+1)(x+1)h} = \lim_{h \rightarrow 0} \frac{-1}{(x+h+1)(x+1)} \\
&= \frac{-1}{(x+1)(x+1)} = -\frac{1}{(x+1)^2}.
\end{aligned}$$

So $f'(x) = -\frac{1}{(x+1)^2}$.

(b) This means that $f'(-\frac{1}{2}) = -\frac{1}{(\frac{1}{2})^2} = -4$, and $f'(5) = -\frac{1}{6^2} = -\frac{1}{36}$.

35. (a) The derivative of $f(x) = 3x^2 + 2x - 10$ is

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(3(x+h)^2 + 2(x+h) - 10) - (3x^2 + 2x - 10)}{h} \\
&= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 10 - 3x^2 - 2x + 10}{h} = \lim_{h \rightarrow 0} \frac{6xh + 3h^2 + 2h}{h} \\
&= \lim_{h \rightarrow 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \rightarrow 0} \frac{6x + 3h + 2}{1} = 6x + 2.
\end{aligned}$$

So $f'(x) = 6x + 2$.

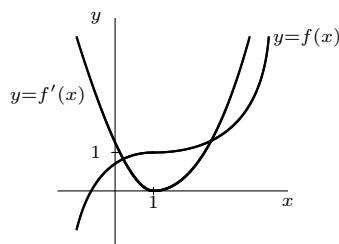
(b) At $a = 1$, we have $f(1) = 3 + 2 - 10 = -5$ and $f'(1) = 6 + 2 = 8$; so the tangent line goes through $(1, -5)$ and has slope 8. Therefore, an equation for the tangent line to the graph at $a = 1$ is

$$y = 8(x - 1) + (-5)$$

$$y = 8(x - 1) - 5$$

$$y = 8x - 13.$$

49. See the book for the graph. A sketch of the function and its derivative is:



To explain: moving left to right, the slope begins large and positive, then comes down towards 0, and is 0 around $x = 1$. Then it starts rising again.

53. See the book for the graph.

- (a) The only value of x in $(0, 3)$ where f is not continuous is $x = 1$.
- (b) The values of $f(x)$ in $(0, 3)$ where f is not differentiable are $x = 1$ (where it is not continuous), and $x = 2$ (where it has a corner).
- (c) Here is a sketch of the graph of $f'(x)$: from 0 to 1, it starts near 0 and rises to about 1. From 1 to 2 it seems to drop from about 0 to about -1 . And from 2 onward it is constant at about 1:

