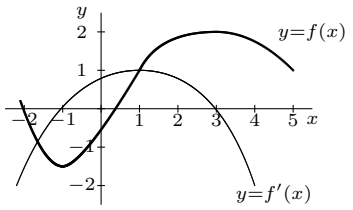
Math 272: Honors Calculus I
Prof. Arturo Magidin
Homework 9
SOLUTIONS

§4.3

7. The thin graph is the given derivative. The other is one possible corresponding function.

   From the sign of the derivative we see that the function is decreasing up to -1; increasing on $[-1,3]$, and decreasing after 3. Looking where the derivative is increasing and where it is increasing, we see the function is concave up up to -1, and concave down after.



23. We have

$$f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 4x$$

and we want to determine the intervals on which f is increasing and the intervals on which it is decreasing.

The function is defined everywhere. The derivative is:

$$f'(x) = x^2 - 5x + 4 = (x-4)(x-1).$$

It is defined everywhere, and is zero at $x = 1$ and $x = 4$.

On $(-\infty, 1)$, we have $f'(x) > 0$ since both factors are positive. On $(1,4)$, we have $f'(x) < 0$; and on $(4,\infty)$, $f'(x)$ is positive.

So $f(x)$ is increasing on $(-\infty, 1]$ and on $[4,\infty)$. It is decreasing on $[1,4]$

29. We want to do the same now with $f(x) = x^2\ln(x^2) + 1$. This is defined for $x \neq 0$.

We have $f'(x) = 2x\ln(x^2) + x^2\left(\frac{2x}{x^2}\right) = 2x(\ln(x^2)+1)$.

This is zero only when $\ln(x^2) = -1$, which happens when $x^2 = e^{-1}$; this means $|x| = \frac{1}{\sqrt{e}}$.

So we have three points where the derivative could change signs: at $x = -\frac{1}{\sqrt{e}}$, at $x = 0$, and at $x = \frac{1}{\sqrt{e}}$.

On $(-\infty, -\frac{1}{\sqrt{e}})$, the derivative is negative; we can see this by plugging in -1, which lies in the interval, and we see we get $-2(\ln(1)+1) = -2 < 0$. On $(-\frac{1}{\sqrt{e}}, 0)$, the derivative is positive; it is negative on $(0, \frac{1}{\sqrt{e}})$, and positive on $(\frac{1}{\sqrt{e}}, \infty)$.

So $f(x)$ is increasing on $[-\frac{1}{\sqrt{e}}, 0)$ and on $[\frac{1}{\sqrt{e}}, \infty)$; and it is decreasing on $(-\infty, -\frac{1}{\sqrt{e}}]$ and on $(0, \frac{1}{\sqrt{e}})$.

36. We now look at $f(x) = x^2\sqrt{9-x^2}$ on $(-3,3)$.

We have

$$\begin{aligned} f'(x) &= 2x\sqrt{9-x^2} + x^2\left(\frac{-2x}{2\sqrt{9-x^2}}\right) \\ &= 2x\sqrt{9-x^2} - \frac{x^3}{\sqrt{9-x^2}} \\ &= \frac{2x(9-x^2)}{\sqrt{9-x^2}} - \frac{x^3}{\sqrt{9-x^2}} \\ &= \frac{18x - 2x^3 - x^3}{\sqrt{9-x^2}} \\ &= \frac{18x - 3x^3}{\sqrt{9-x^2}} = \frac{3x(6-x^2)}{\sqrt{9-x^2}} \\ &= \frac{3x(\sqrt{6}-x)(\sqrt{6}+x)}{\sqrt{9-x^2}}. \end{aligned}$$

This always defined on $(-3, 3)$; it is zero at $x = 0$, at $x = -\sqrt{6}$, and at $x = \sqrt{6}$.

If $-\sqrt{6} < x < \sqrt{6}$, then $6 - x^2 > 0$; and $6 - x^2 < 0$ otherwise. The denominator is always positive; and the sign of $3x$ depends on the sign of x. So we have:

On $(-3, -\sqrt{6})$, both factors iin the numerator are negative, so $f'(x) > 0$. On $(-\sqrt{6}, 0)$, we have $f'(x) < 0$. On $(0, \sqrt{6})$ we have $f'(x) > 0$; and on $(\sqrt{6}, 3)$ we have $f'(x) < 0$.

So $f(x)$ is increasing on $(-3, -\sqrt{6}]$ and on $[0, \sqrt{6}]$. And $f(x)$ is decreasing on $[-\sqrt{6}, 0]$ and on $[\sqrt{6}, 3)$.

40. If $f(x) = x^2 - 2\ln x$, then the domain is $(0, \infty)$.

The derivative is $f'(x) = 2x - \frac{2}{x}$. This is defined on all of $(0, \infty)$. The critical points occur when $2x = \frac{2}{x}$, which is when $x^2 = 1$. Since x must be positive, the only location of a critical point is $x = 1$.

On $(0, 1)$, the derivative $f'(x) = 2x - \frac{2}{x}$ is negative. And on $(1, \infty)$, we have $f'(x) > 0$.

So $f(x)$ is increasing on $[1, \infty)$ and decreasing on $(0, 1]$.

47. The function is $f(x) = x\sqrt{4-x^2}$ on $[-2, 2]$.

(a) To find the critical points of $f(x)$, we compute the derivative:

$$\begin{aligned} f'(x) &= \sqrt{4-x^2} + x\left(\frac{-2x}{2\sqrt{4-x^2}}\right) \\ &= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \\ &= \frac{4-x^2}{\sqrt{4-x^2}} - \frac{x^2}{\sqrt{4-x^3}} \\ &= \frac{4-2x^2}{\sqrt{4-x^2}}. \end{aligned}$$

This is defined everywhere in $(-2, 2)$, but undefined at $x = -2$ and $x = 2$. These are the endpoints, so they are not relevant for local extremes.

The other critical points occur when $f'(x) = 0$, which happens when $4 - 2x^2 = 0$. This means $x^2 = 2$. So we get $x = \sqrt{2}$ and $x = -\sqrt{2}$.

So the critical points in $[-2, 2]$ are $x = \sqrt{2}$ and $x = -\sqrt{2}$.

(b) The denominator of the derivative is always positive. So the sign of the derivative only depends on the sign of $4 - 2x^2 = 2(2 - x^2) = 2(\sqrt{2} - x)(\sqrt{2} + x)$.

Before $-\sqrt{2}$, the derivative is negative; between $-\sqrt{2}$ and $\sqrt{2}$ we have $f'(x) > 0$. And for $x > \sqrt{2}$, the derivative is negative again.

By the First Derivative Test: at $x = -\sqrt{2}$, the derivative changes from negative to positive, so we have a local minimum. And at $x = \sqrt{2}$ the derivative changes from positive to negative, so the function has a local maximum. The local minimum is $f(-\sqrt{2}) = -\sqrt{2}\sqrt{2} = -2$. The local maximum is $f(\sqrt{2}) = \sqrt{2}\sqrt{2} = 2$.

(c) For the absolute extremes we also compare with the values at the endpoints. We have $f(2) = 2\sqrt{4 - 2^2} = 0$, and $f(-2) = -2\sqrt{4 - (-2)^2} = 0$. So the absolute maximum of $f(x)$ on $[-2, 2]$ is 2, achieved at $x = \sqrt{2}$, and the absolute minimum is -2, achieved at $x = -\sqrt{2}$.

53. Now we have $f(x) = \sqrt{x} \ln x$ on $(0, \infty)$.

(a) The derivative is $f'(x) = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}$. This is defined on all of $(0, \infty)$.

The stationary points, where $f'(x) = 0$, are the ones where:

$$\begin{aligned} \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} &= 0 \\ \frac{\ln x}{2\sqrt{x}} &= -\frac{\sqrt{x}}{x} \\ \ln x &= -2\frac{x}{x} \\ \ln x &= -2 \\ x &= e^{-2}. \end{aligned}$$

So the only critical point is $x = e^{-2}$

(b) For large values of x, $f'(x) > 0$. So the derivative is positive on (e^{-2}, ∞). If we take, say, $x = e^{-4}$ to test for the sign of $f'(x)$ on $(0, e^{-2}$, we have:

$$f'(e^{-4}) = \frac{-4}{2e^{-2}} + \frac{e^{-2}}{e^{-4}} = -2e^2 + e^2 = -e^2 < 0.$$

So the derivative changes from negative to positive at $x = e^{-2}$, which means that $f(x)$ has a local minimum at $x = e^{-2}$.

(c) Since this is the only local extreme on $(0, \infty)$, we conclude that the function has an absolute extreme at $x = e^{-2}$, with value $f(e^{-2}) = e^{-1} \ln(e^{-2}) = -\frac{2}{e}$.

The function has no absolute maximum: note that $\lim_{x \to \infty} f(x) = \infty$, so there can be no absolute maximum.

77. Let $f(x) = x^3 - 3x^2$.

To find the critical points, we compute $f'(x) = 3x^2 - 6x = 3x(x - 2)$. So the critical points are $x = 0$ and $x = 2$.

To apply the second derivative test, we calculate $f''(x) = 6x - 6$. We have $f''(0) = -6 < 0$ and $f''(2) = 6 > 0$.

Since $f'(0) = 0$ and $f''(0) < 0$, the Second Derivative Test tells us that f has a local maximum at $x = 0$. Because $f'(2) = 0$ and $f''(2) > 0$, it tells us that f has a local minimum at $x = 2$.

81. For $f(x) = e^x(x-7)$, we have:

$$f'(x) = e^x(x-7) + e^x = e^x(x-7+1) = e^x(x-6)$$
$$f''(x) = e^x(x-6) + e^x = e^x(x-5).$$

The only critical point occurs at $x = 6$; and there, $f''(x) > 0$, so the function has a local minimum at $x = 6$.

§4.4

19. We have the function $f(x) = x^4 - 6x^2$.

The function is defined everywhere. It is an even function: $f(-x) = (-x)^4 - 6(-x)^2 = x^4 - 6x^2 = f(x)$. So the graph is symmetric about the y-axis.

To find the shape of the graph, we take derivatives:

$$f'(x) = 4x^3 - 12x = 4x(x^2-3) = 4x(x+\sqrt{3})(x-\sqrt{3}),$$
$$f''(x) = 12x^2 - 12 = 12(x^2-1) = 12(x+1)(x-1).$$

The critical points are at $x = 0$, $x = -\sqrt{3}$ and at $x = \sqrt{3}$. The second derivative is 0 at $x = -1$ and at $x = 1$. Here is a sign chart:

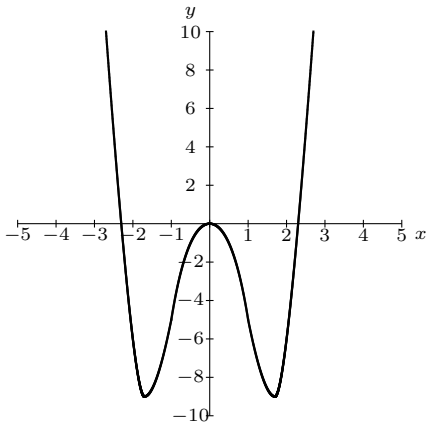
	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$-$	$+$	$+$	$-$	$-$	$+$
$f''(x)$	$+$	$+$	$-$	$-$	$+$	$+$
$f(x)$						

We have a local minimum at $x = \sqrt{3}$ and at $x = -\sqrt{3}$, with value $f(\sqrt{3}) = 9 - 18 = -9$. We have a local maximum at $x = 0$ with value $f(0) = 0$. We have a points of inflection at $x = 1$ and at $x = -1$.

The y-intercept is $f(0) = 0$. The x intercepts occur when $f(x) = x^4 - 6x^2 = 0$; This occurs when $x^2(x^2-6) = 0$, which is when $x = 0$ and when $x = -\sqrt{6}$ and $x = \sqrt{6}$.

There are no vertical asymptotes. And $\lim_{x\to\infty} f(x) = \infty$.

Putting it all together: on $[0, \infty)$, the function has interecepts at $(0, 0)$ and at $(\sqrt{6}, 0)$. It has a local maximum of 0 at 0, and a local minimum of -9 at $x = \sqrt{3}$. IIt has a point of inflection at $x = 1$ (the point on the graph is $(1, -5)$, since $f(1) = -5$). And it goes to infinty as $x \to \infty$. The function is even. So we get the following picture:



28. For $f(x) = 3\sqrt{x} - x^{3/2} = \sqrt{x}(3-x)$.

The domain of the function is $[0, \infty)$, so there are no symmetries to find. It is not periodic.

We have:

$$f'(x) = \frac{3}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{3}{2}x^{-1/2}(1-x).$$
$$f''(x) = -\frac{3}{4}x^{-3/2} - \frac{3}{4}x^{-1/2} = -\frac{3}{4}x^{-3/2}(1+x).$$

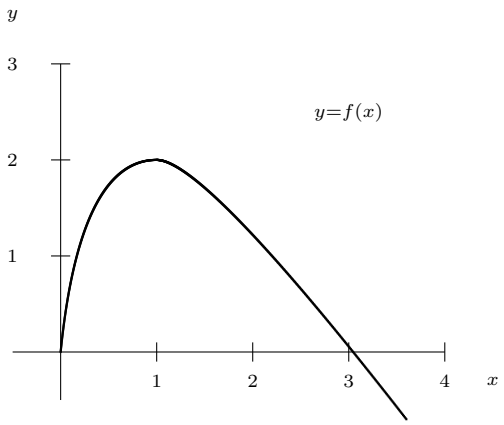
So the critical points on $[0, \infty)$ are $x = 0$, which is an endpoint; and $x = 1$. On the other hand, the second derivative is always negative, so $f(x)$ is concave down on $[0, \infty)$, with no points of inflection.

The derivative is positive on $(0, 1)$ and negative on $(1, \infty)$; so $f(x)$ has a local maximum at $x = 1$, with value $f(1) = 2$.

The interecepts are $f(0) = 0$, the origin. And the points where $f(x) = 0$. Writing $f(x) = \sqrt{x}(3-x)$, we see that the other x-intercept occurs at $x = 3$.

For the end behavior, we note that $\lim_{x\to\infty} f(x) = -\infty$, since the $-x^{3/2}$ term will dominate.

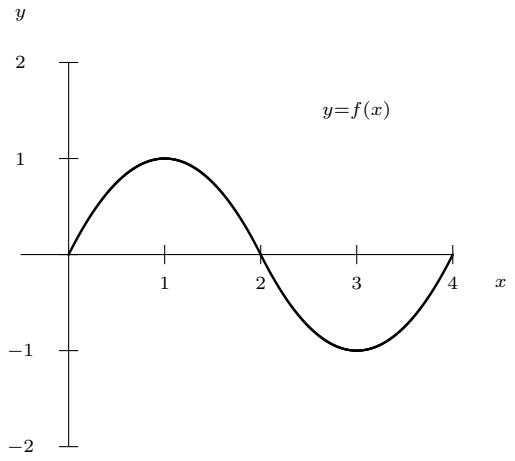
So we have the following sketch:



47. See the book for the graphs of $f'(x)$ and $f''(x)$ The function is increasing on $[0, 1]$, decreasing on $[1, 3]$, and increasing on $[3, 4]$. It has a local maximum at $x = 1$ and a local minimum at $x = 3$.

The function is concave down on $(0, 2)$ and concave up on $(2, 4)$. It has a point of inflection at $x = 2$.

A possible graph with $f(0) = 0$ is:



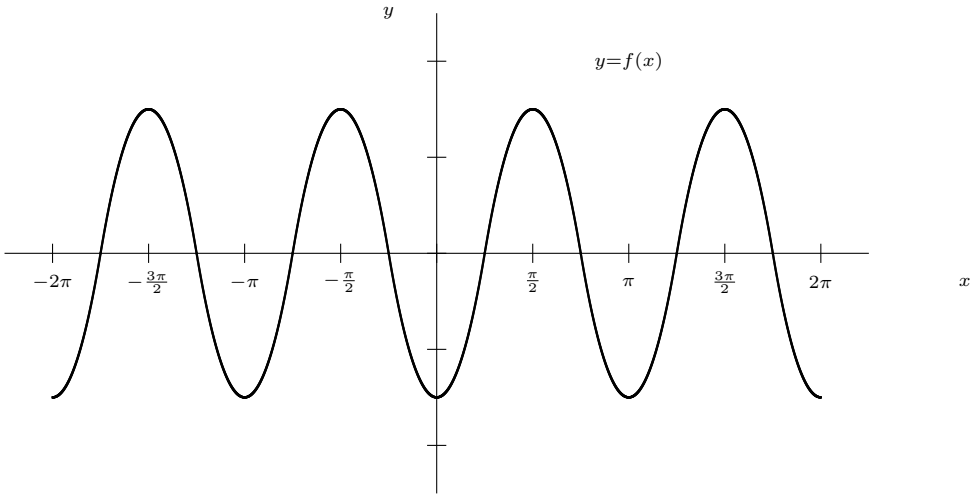
57. Suppose the derivative of $f(x)$ is $f'(x) = 10\sin(2x)$, on $[-2\pi, 2\pi]$.

The derivative is 0 at $\pm 2\pi$, $\pm\frac{3\pi}{2}$, $\pm\pi$, $\pm\frac{\pi}{2}$, and at $x = 0$.

It is positive on $(-2\pi, -3\frac{\pi}{2})$, $(-\pi, -\frac{\pi}{2})$, $(0, \frac{\pi}{2})$, and $(\frac{3\pi}{2}, 2\pi)$. It is negative on the remaining portions.

There are local maxima at $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$. There are local minima at $x = -\pi$, $x = 0$, and $x = \pi$.

Here's an example of such a function:



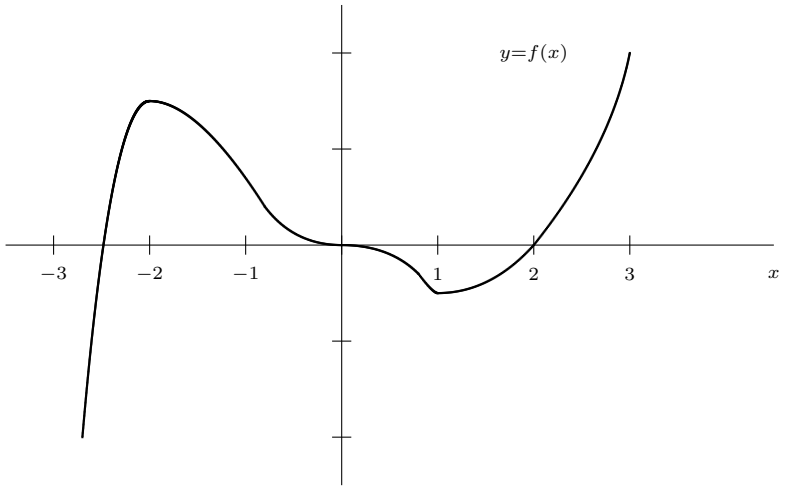
59. If $f'(x) = x^2(x+2)(x-1)$, we see that there are critical points at $x = 0$, $x = 1$, and $x = -2$.

The derivative is positive on $(1, \infty)$ and on $(-\infty, -2)$. It is negative on $(-2, 0)$ and on $(0, 1)$.

So the function has a relative maximum at $x = -2$ and a local minimum at $x = 1$; it has a horizontal tangent but no local extreme at $x = 0$.

The function is decreasing on $[-2, 1]$; it is increasing on $(-\infty, -2]$ and on $[1, \infty)$.

Here is one possible graph for $f(x)$: y



§4.5

19. If the side of the base has length x, and the box has height h, then the volume is $V = x^2h$, and the surface area is $S = 2x^2 + 4xh$ (the base and top have area x^2, the sides have area xh).

    Since we want the volume to be 8 m^3, we must have $x^2h = 8$, so $h = \frac{8}{x^2}$. Substituting into the surface, we get:

$$S(x) = 2x^2 + 4x\left(\frac{8}{x^2}\right) = 2x^2 + \frac{32}{x}.$$

    The interval of interest is $(0, \infty)$, and we want to find the minimum.

    Note that $\lim_{x\to\infty} S(x) = \infty$ and $\lim_{x\to 0^+} S(x) = \infty$. So there should be a minimum.

    The critical points are given by $S'(x) = 4x - \frac{32}{x^2}$. This is zero when

$$\begin{aligned} 4x - \frac{32}{x^2} &= 0 \\ 4x &= \frac{32}{x^2} \\ x^3 &= 8 \\ x &= 2. \end{aligned}$$

    Note that $S''(x) = 4 + \frac{64}{x^3}$, so $S''(2) > 0$; thus, the function has a local minimum at $x = 2$. Since this is the only relative extreme on the interval $(0, \infty)$, it is in fact an absolute minimum.

    The question asks for the dimensions of the box; we have concluded that the base is 2 meters by 2 meters. From our formula, we see that $h = \frac{8}{x^2} = \frac{8}{4} = 2$ meters.

    So the dimensions of the box of least surface area are $2 \times 2 \times 2$ meters: a cube.

21. Let the length of the side of the base be x, and the height h. The volume is fixed at $16 = x^2h$, so $h = \frac{16}{x^2}$.

What is the total cost of a box with base length x and height h? The base costs twice as much per square foot as the sides; and the top costs half as much. If the cost per square foot of the sides is c (constant), then the total cost is: $(2c)x^2 + c(4xh) + \frac{c}{2}x^2$. Plugging in the value for h and simplifying, we get:

$$\begin{aligned} f(x) &= 2cx^2 + 4cxh + \frac{c}{2}x^2 \\ &= \left(2c + \frac{c}{2}\right)x^2 + 4cx\left(\frac{16}{x^2}\right) \\ &= \frac{5c}{2}x^2 + \frac{64c}{x} \\ &= c\left(\frac{5x^2}{2} + \frac{64}{x}\right). \end{aligned}$$

We don't actually care about the value of c; it is positive, so it will simply scale the total. So we can set it to 1 and proceed.

We are working on the interval $(0, \infty)$. Note that $\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f(x) = \infty$, so there will be a minimum.

We have: $f'(x) = 5x - \frac{64}{x^2}$. This is zero when $x^3 = \frac{64}{5}$, which yields $x = \frac{4}{\sqrt[3]{5}}$.

Since $f''(x) = 5 + \frac{128}{x^3}$, the value at the critical point is positive. So $f(x)$ has a local minimum at $\frac{4}{\sqrt[3]{5}}$. Since this is the only local extreme on $(0, \infty)$, it is an absolute minimum.

Since $h = \frac{16}{x^2} = \frac{16\sqrt[3]{25}}{16} = \sqrt[3]{25}$, we conclude that the box-shaped crate of least cost has a $\frac{4}{\sqrt[3]{5}}$ by $\frac{4}{\sqrt[3]{5}}$ ft base, and a height of $\sqrt[3]{25}$ feet.

35. Let r and h be the radius and height of the can. Since the volume is 354 cubic centimeters, we have

$$V = 354 = \pi r^2 h, \qquad \text{or} \qquad h = \frac{354}{\pi r^2}.$$

The top and bottom of the can have area of πr^2; the side has area $(2\pi r)h$.

(a) If the top and bottom cost the same to make, then the total cost is proportional to the surface area. So we just need to figure out the surface area and minimize it. We have:

$$S = 2\pi r^2 + 2\pi r h = 2\pi\left(r^2 + r\left(\frac{354}{\pi r^2}\right)\right) = 2\pi\left(r^2 + \frac{354}{\pi r}\right).$$

We want to minimize over $(0, \infty)$.

The derivative is

$$S'(r) = 2\pi\left(2r - \frac{354}{\pi r^2}\right).$$

So the critical point has

$$\begin{aligned} 2r - \frac{354}{\pi r^2} &= 0 \\ 2r &= \frac{354}{\pi r^2} \\ r^3 &= \frac{354}{2\pi} \\ r &= \sqrt[3]{\frac{354}{2\pi}}. \end{aligned}$$

Note that $S''(r) = 2\pi\left(2 + \frac{690}{\pi r^3}\right)$.

Using the Second Derivative Test we see that it is a local minimum, and since it is the only local extreme on $(0, \infty)$, it is an absolute minimum. The corresponding value of h is

$$h = \frac{354}{\pi r^2} = \frac{354r}{\pi r^3} = \frac{354r}{\pi}\left(\frac{2\pi}{354}\right) = 2r.$$

So the diameter of the can equals its height. The can would look like a square seen from the side.

(b) If the top and bottom of the can have "double thickness", then the cost function is not simply the sufrace area, but instead it is given by

$$F(r) = 2\pi\left(2r^2 + \frac{354}{\pi r}\right) = 4\pi\left(r^2 + \frac{177}{\pi r}\right).$$

Proceeding as above, we have

$$F'(r) = 4\pi\left(2r - \frac{177}{\pi r^2}\right),$$

so the only critical point lies at $r = \sqrt[3]{\frac{177}{2\pi}}$. It is an absolute minimum, and the corresponding height is

$$h = \frac{354r}{\pi r^3} = \frac{345r}{\pi}\left(\frac{2\pi}{177}\right) = 4r.$$

So now the height is four times the radius (twice the diameter), which matches with the dimensions given in the problem.

37. If we let x be the length of the flower garden and y the height, then we want $xy = 30$. Once we include the borders, the whole thing will have area

$$A = (x + 4)(y + 2) = xy + 4y + 2x + 8.$$

Substituting $y = \frac{30}{x}$, we obtain the function

$$A(x) = 30 + \frac{120}{x} + 2x + 8 = 2x + \frac{120}{x} + 38.$$

The domain is $(0, \infty)$.

The critical points are given by $A'(x) = 2 - \frac{120}{x^2}$. This gives $x^2 = 60$, so $x = \sqrt{60} = 2\sqrt{15}$.

The second derivative is $A''(x) = \frac{240}{x^3}$, so $A''(2\sqrt{15}) > 0$. Thus, we have a local minimum there. Since it is the only local extreme, it is an absolute minimum.

When $x = 2\sqrt{15}$, we have $y = \frac{30}{x} = \frac{2(15)}{2\sqrt{15}} = \sqrt{15}$.

Thus, the dimensons of the flower garden that minimze the combined area are $2\sqrt{15}$ meters by $\sqrt{15}$ meters, with the longer side having the 1 meter grass border.

46. Let x be the number of tickets sold; we will have $20 \le x \le 70$. The cost per person is $30 - \frac{1}{4}x$. The fixed expenses are 200. So we will receive $30 - \frac{1}{4}x$ dollars for each of the x tickets sold, and our total profit will be

$$P(x) = x\left(30 - \frac{1}{4}x\right) - 200 = 30x - \frac{1}{4}x^2 - 200.$$

The critical points are given by $P'((x) = 30 - \frac{x}{2}$, that is, $x = 60$.
Evaluating at $x = 20$, $x = 60$, and $x = 70$, we have:

$$P(20) = 30(20) - \frac{1}{4}(20)^2 - 200 = 600 - 100 - 200 = 300;$$
$$P(60) = 30(60) - \frac{1}{4}(60)^2 - 200 = 1800 - 900 - 200 = 700;$$
$$P(70) = 30(70) - \frac{1}{4}(70)^2 - 200 = 2100 - 1225 - 200 = 675.$$

So the maximum profit is achieved when we sell 60 tickets.