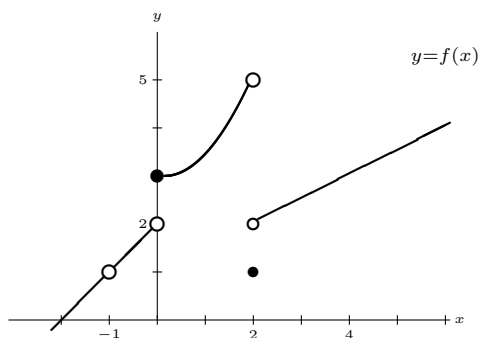


CHAPTER 2 TEST

SOLUTIONS

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1. Use the following figure to give values of the following limits (if they exist), and values of the function (if they are defined). If they do not exist or are undefined, say so explicitly. You do not need to provide an explanation, just the answer. (1 point each, 12 points total)



Answers: These follow by looking at the graph.

- (i) $\lim_{x \rightarrow 2^+} f(x) = 2$.
 - (ii) $\lim_{x \rightarrow 2^-} f(x) = 5$.
 - (iii) $\lim_{x \rightarrow 2} f(x)$ does not exist (one sided limits are different from each other).
 - (iv) $f(2) = 1$.
 - (v) $\lim_{x \rightarrow 0^+} f(x) = 3$.
 - (vi) $\lim_{x \rightarrow 0^-} f(x) = 2$.
 - (vii) $\lim_{x \rightarrow 0} f(x)$ does not exist (one-sided limits are different).
 - (viii) $f(0) = 3$.
 - (ix) $\lim_{x \rightarrow -1^+} f(x) = 1$.
 - (x) $\lim_{x \rightarrow -1^-} f(x) = 1$.
 - (xi) $\lim_{x \rightarrow -1} f(x) = 1$.
 - (xii) $f(-1)$ is undefined.
2. In this page and the next are five limits. Use valid limit laws and algebraic manipulations to determine their value, or that they do not exist. You may **not** use sampling or a graphing calculator to try to determine the value of the limits. If the limit does not exist and is ∞ or $-\infty$, you must say so explicitly. If a limit does not exist but does not equal ∞ nor $-\infty$, then you must explicitly write that the limit DOES NOT EXIST. (3 points each, 15 points total)

(i) $\lim_{x \rightarrow -2} \frac{2x^2 - 5}{x^2 + 4}$

Answer. This is a rational function that is defined at $x = -2$, so the value of the limit can be found by simply plugging in $x = -2$:

$$\lim_{x \rightarrow -2} \frac{2x^2 - 5}{x^2 + 4} = \frac{2(-2)^2 - 5}{(-2)^2 + 4} = \frac{8 - 5}{4 + 4} = \frac{3}{8}.$$

(ii) $\lim_{x \rightarrow -\infty} e^x$

Answer. We saw in class that $\lim_{x \rightarrow -\infty} e^x = 0$. Just saying it equals 0 is sufficient here, since it was a limit we just stated in class.

(iii) $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$

Answer. If we try evaluating by plugging in $x = 9$, we get $\frac{0}{0}$, which is an indeterminate limit. We multiply and divide by the conjugate of the numerator to get rid of the radical, cancel, and evaluate:

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} = \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} \\ &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6}. \end{aligned}$$

(iv) $\lim_{x \rightarrow 2^+} \frac{x - 3}{x^2 - 4}$

Answer. This limit does not exist, since the numerator approaches -1 and the denominator approaches 0. But we can be more precise:

For x very close to and larger than 2, x^2 will be very close and larger than 4, so the denominator will be positive and close to 0. Dividing a number close to -1 by a small positive number will yield a large negative number, and the closer x gets 2, the larger this quotient will be (while remaining negative). So

$$\lim_{x \rightarrow 2^+} \frac{x - 3}{x^2 - 4} = -\infty.$$

(v) $\lim_{x \rightarrow \infty} \left(1000x^2 - \frac{1}{350}x^3 \right)$

Answer. The leading term here is $-\frac{1}{350}x^3$, so the polynomial will have the same end behavior as this term.

$$\lim_{x \rightarrow \infty} \left(1000x^2 - \frac{1}{350}x^3 \right) = \lim_{x \rightarrow \infty} x^3 \left(\frac{1000}{x} - \frac{1}{350} \right) = -\infty.$$

3. Use the definition (the “continuity checklist”) to determine whether the function

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{if } x \neq 1, \\ 3 & \text{if } x = 1. \end{cases}$$

is continuous at $x = 1$. State your conclusion explicitly, justifying it with the prior work. (10 points)

Answer.

Is $f(x)$ defined at $x = 1$? Yes: $f(1) = 3$.

Does $\lim_{x \rightarrow 1} f(x)$ exist? Yes:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x + 1}{1} = \frac{1 + 1}{1} = 2.$$

Finally, is the value of the limit equal to the value of the function at the point? **No**. Here,

$$\lim_{x \rightarrow 1} f(x) = 2 \neq 3 = f(1).$$

Since the third item in the checklist fails, we conclude that $f(x)$ is **not continuous** at $x = 1$.

4. Use the Intermediate Value Theorem to verify that the equation

$$2x^3 + x - 2 = 0$$

has at least one solution in the interval $(-1, 1)$.

To solve this problem you must explicitly:

- (i) State the function $f(x)$ to which you will apply the Intermediate Value Theorem. *(1 point)*
- (ii) State the closed interval $[a, b]$ on which $f(x)$ is continuous, and say a few words about how you know the function is continuous on that interval. *(2 points)*
- (iii) Identify the value k that you are targetting, and verify that it lies between $f(a)$ and $f(b)$. *(3 points)*
- (iv) State the conclusion of the Intermediate Value Theorem as applied to the data from (i)–(iii), and how it solves the problem. *(1 point)*

(Total: 7 points)

Answer.

- (i) There are many possibilities, but the simplest one is to take $f(x) = 2x^3 + x - 2$.
- (ii) We consider the interval $[-1, 1]$. The function $f(x)$ is continuous on this interval because $f(x)$ is a polynomial and hence is continuous everywhere.
- (iii) We want to find a point c where $f(c) = 0$. Note that 0 lies between $f(-1) = -5$ and $f(1) = 1$.
- (iv) The Intermediate Value Theorem, applied to the data above, tells us that there is at least one point c , with $-1 < c < 1$, and where $f(c) = 0$. This means a point where $2c^3 + c - 2 = 0$. This is precisely a solution to the equation that is in the interval $(-1, 1)$, the conclusion we want.

5. Let $f(x)$ be the following function:

$$f(x) = \frac{3x^2 - 7x + 12}{5x - 4 - x^2}.$$

- (i) Find the domain of $f(x)$. *(2 points)*

Answer. We have

$$f(x) = \frac{3x^2 - 7x + 12}{5x - 4 - x^2} = \frac{3x^2 - 7x + 12}{-(x^2 - 5x + 4)} = \frac{3x^2 - 7x + 12}{-(x - 4)(x - 1)}.$$

The rational function is defined everywhere that the denominator does not equal 0, which are the values of x different from 1 and 4. So the domain of $f(x)$ is

$$(-\infty, 1) \cup (1, 4) \cup (4, \infty).$$

- (ii) Find the horizontal asymptote of $y = f(x)$ as $x \rightarrow -\infty$, or explain why no horizontal asymptote exists. (4 points)

Answer. For this we do not need the factorization: this essentially asks for the limit of $f(x)$ as $x \rightarrow -\infty$. We have:

$$\begin{aligned}\lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{3x^2 - 7x + 12}{5x - 4 - x^2} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}(3x^2 - 7x + 12)}{\frac{1}{x^2}(5x - 4 - x^2)} \\ &= \lim_{x \rightarrow -\infty} \frac{3 - \frac{7}{x} + \frac{12}{x^2}}{\frac{5}{x} - \frac{4}{x^2} - 1} = \frac{3 - 0 + 0}{0 - 0 - 1} \\ &= -3.\end{aligned}$$

So the horizontal asymptote of $y = f(x)$ as $x \rightarrow -\infty$ is the line $y = -3$.