

CHAPTER 3 TEST

SOLUTIONS

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1. Find $f^{(5)}(x)$, the **fifth derivative** of $f(x)$, where

$$f(x) = \sin(x) - 2x^3 + \ln(x) + e^4.$$

(7 points)

Answer. The two things to be careful about are (i) that e^4 is a constant; and (ii) that for later derivatives, it is easier to write $(\ln x)' = \frac{1}{x}$ as x^{-1} , so that we apply the power rule instead of a quotient or reciprocal rule for the second and higher derivatives. We have:

$$\begin{aligned} f(x) &= \sin(x) - 2x^3 + \ln(x) + e^4 \\ f'(x) &= \cos(x) - 6x^2 + x^{-1} \\ f''(x) &= -\sin(x) - 12x - x^{-2} \\ f'''(x) &= -\cos(x) - 12 + 2x^{-3} \\ f^{(4)}(x) &= \sin(x) - 6x^{-4} \\ f^{(5)}(x) &= \cos(x) + 24x^{-5} = \cos(x) + \frac{24}{x^5}. \end{aligned}$$

2. This page contains two functions; the next page contains three. For each of them, find the derivative $f'(x)$. Your final answer should contain no derivative left indicates, no complex fractions (fractions of fractions), and no easy simplifications left (no expressions like $2 + 3$ or $3x - 6x$), but you do not need to write it in simplest possible terms. (3 points each, 15 points total)

- (i) $f(x) = \arccos(x)$

Answer. This is one of our basic functions, so we can just write down the answer (having, presumably, memorized it):

$$f'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

- (ii) $f(x) = x^3 \ln(x)$

Answer. This is a Product Rule, with a simplification:

$$f'(x) = (x^3)' \ln(x) + x^3 (\ln x)' = 3x^2 \ln(x) + x^3 \left(\frac{1}{x} \right) = 3x^2 \ln(x) + x^2 = x^2(3 \ln(x) + 1).$$

- (iii) $f(x) = \frac{\tan x}{x+1}$

Answer. A Quotient Rule:

$$f'(x) = \frac{(x+1)(\tan x)' - (\tan x)(x+1)'}{(x+1)^2} = \frac{(x+1) \sec^2(x) - \tan x}{(x+1)^2}.$$

- (iv) $f(x) = e^{\sec x}$.

Answer. This is a Chain Rule:

$$f'(x) = e^{\sec x} (\sec x)' = \sec(x) \tan(x) e^{\sec x}.$$

$$(v) f(x) = \frac{e^x + e^{-x}}{2}$$

Answer. This has a Chain Rule (for the derivative of e^{-x}), but we do **not** need to use a Quotient Rule if we just take the $\frac{1}{2}$ out:

$$f'(x) = \left(\frac{1}{2}(e^x + e^{-x}) \right)' = \frac{1}{2}(e^x + e^{-x})' = \frac{1}{2}(e^x + (-x)'e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \frac{e^x - e^{-x}}{2}.$$

3. Use implicit differentiation to find the equation of the tangent line to the curve defined implicitly by:

$$x^3 + y^3 = 2xy$$

at the point $(1, 1)$. Write the equation of the tangent line in the form $y = mx + b$. You do **not** need to give a formula for y' .

Do not attempt to solve for y first; you must use implicit differentiation. (10 points)

Answer. Note that the point is on the curve: plugging in, we have $(1)^3 + (1)^3 = 2(1)(1)$, which is correct.

We can take derivatives on both sides, remembering to treat y as a function, not a variable. We also do not need a full formula for y' : we just need the value at $(1, 1)$, so we can plug that in before solving for y' . That simplifies the calculations. We have:

$$\begin{aligned} x^3 + y^3 &= 2xy \\ \frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(2xy) \\ 3x^2 + 3y^2y' &= 2((x)'y + x(y)') \\ 3x^2 + 3y^2y' &= 2(y + xy'). \end{aligned}$$

Plugging in $x = 1$, $y = 1$, and then solving for y' we have

$$\begin{aligned} 3(1)^2 + 3(1)^2y' &= 2(1 + (1)y') \\ 3 + 3y' &= 2 + 2y' \\ 3y' - 2y' &= 2 - 3 \\ y' &= -1. \end{aligned}$$

So the tangent at $(1, 1)$ has slope -1 . To find the equation of the tangent line, we use the point-slope formula, with point $(1, 1)$ and slope -1 to get:

$$\begin{aligned} y - 1 &= -1(x - 1) \\ y - 1 &= -x + 1 \\ y &= -x + 2. \end{aligned}$$

REMARK. If we solve for y' to obtain a general formula before plugging in $x = y = 1$, we would get:

$$\begin{aligned} 3x^2 + 3y^2y' &= 2y + 2xy' \\ 3y^2y' - 2xy' &= 2y - 3x^2 \\ (3y^2 - 2x)y' &= 2y - 3x^2 \\ y' &= \frac{2y - 3x^2}{3y^2 - 2x}. \end{aligned}$$

Plugging in $x = y = 1$ we get $y' = \frac{-1}{1} = -1$; and then proceed as above.

4. Suppose the position of an object moving horizontally along a line after t seconds is given by the following function $s(t)$, measured in feet, with $s > 0$ representing positions to the right of the origin:

$$s(t) = 2t^2 - 9t + 12, \quad 0 \leq t \leq 3.$$

- (i) Determine the average velocity of the object between times $t = 0$ and $t = 3$. Don't forget the units. (2 points)

Answer. The average velocity is

$$\begin{aligned} v_{\text{ave}} &= \frac{s(3) - s(0)}{3 - 0} = \frac{(18 - 27 + 12) - (0 - 0 + 12)}{3} \\ &= \frac{3 - 12}{3} = -\frac{9}{3} = -3 \text{ ft/sec.} \end{aligned}$$

So the average velocity is -3 ft/sec.

- (ii) Determine the instantaneous velocity of the object at $t = 1$. Don't forget the units. (2 points)

Answer. The instantaneous velocity is given by $v(t) = s'(t)$. We have $s'(t) = 4t - 9$, so the instantaneous velocity at $t = 1$ is $v(1) = 4(1) - 9 = -5$ ft/sec.

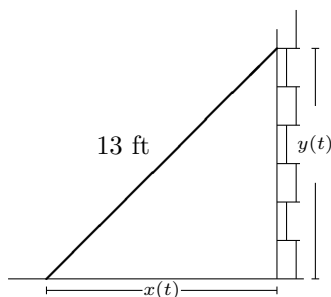
- (iii) Determine the intervals on which the object is speeding up. (4 points)

Answer. The object is speeding up when the instantaneous velocity and the instantaneous acceleration have the same sign. The acceleration is $a(t) = v'(t) = s''(t) = 4$ feet/sec². This is always positive, so we are asking when is the velocity positive.

The velocity is $v(t) = s'(t) = 4t - 9$. So $4t - 9 > 0$ when $4t > 9$, which happens when $t > \frac{9}{4}$. So the object is speeding up on the time interval $(\frac{9}{4}, 3]$.

5. A 13 foot ladder is leaning against a vertical wall. The foot of the ladder starts slipping away from the wall at a rate of $\frac{1}{2}$ ft/sec. The top of the ladder remains on the wall, but starts sliding down. How fast is the top of the ladder sliding down the wall the instant that the foot of the ladder is 5 ft from the wall? (10 points)

Answer. Let $x(t)$ be the distance from the foot of the ladder to the wall at time t , in feet, and $y(t)$ the distance from the top of the ladder to the ground at time t , also in feet. We measure t in seconds:



We are told that $\frac{dx}{dt} = \frac{1}{2}$ ft/sec (positive because it is moving away from the wall, so x is increasing). We want to know the value of $\frac{dy}{dt}$ when $x = 5$.

An equation that connects $x(t)$ and $y(t)$ is

$$13^2 = x^2 + y^2.$$

Differentiating implicitly, we get

$$0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$

We want to plug in the values of x and $\frac{dx}{dt}$ and solve for $\frac{dy}{dt}$; but we also need the value of y when $x = 5$.

If $13^2 = x^2 + y^2$, then at $x = 5$ we have $169 = 25 + y^2$, so $y^2 = 144$. Therefore, $y = 12$. Plugging in, we have

$$\begin{aligned} 0 &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ 0 &= 2(5) \left(\frac{1}{2} \right) + 2(12) \frac{dy}{dt} \\ 0 &= 5 + 24 \frac{dy}{dt} \\ -5 &= 24 \frac{dy}{dt} \\ -\frac{5}{24} &= \frac{dy}{dt}. \end{aligned}$$

Negative because the ladder is sliding down.

So when the foot of the ladder is 5 feet from the wall, the top of the ladder is sliding down the wall at a rate of $\frac{5}{24}$ feet per second.