MATH 272-004 — FALL 2026

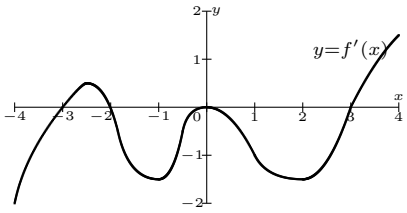
**CHAPTER 4 TEST** SOLUTIONS

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1. The following sketch represents the graph of **the derivative** $f'(x)$ of a function that is defined and continuous at every point in the interval $[-4, 4]$. Use this graph to answer the following questions about the function $f(x)$.

   For parts (a)–(d), you do not need to explain or justify, just provide the answer. For part (e), you must justify your answer as instructed.

   **Note well:** The questions are about $f(x)$; the graph you are seeing is the graph of $f'(x)$, **not** of $f(x)$.




   **Answer.** I provide explanations for all parts, even though you are only required to explain your responses for part (e).

   (a) On what intervals is $f(x)$ increasing? *(2 points)*

   **Answer.** $f(x)$ is increasing on the (closed) intervals where $f'(x)$ is positive. So here it is on $[-3, -2]$ and $[3, 4]$.

   (If you answered $[3, \infty)$ instead of $[3, 4]$, I will mark it correct, even though the problem states we are dealing with a function that is defined only on $[-4, 4]$).

   (b) On what intervals is $f(x)$ concave up? *(2 points)*

   **Answer.** $f(x)$ is concave up on the (open) intervals where $f'(x)$ is increasing. Here, it is $[-4, -2.5)$, $(-1, 0)$, and $(2, 4]$.

   (c) At which points does $f(x)$ have points of inflection? *(2 points)*

   **Answer.** The points of inflection of $f(x)$ correspond to the local extremes of $f'(x)$; so $x = -2.5$, $x = -1$, $x = 0$, and $x = 2$.

   (d) What are the critical points of $f(x)$? *(2 points)*

   **Answer.** The critical points of $f(x)$ are the points where $f'(x)$ is undefined or equal to 0. Here, they are $x = -3$, $x = -2$, $x = 0$, and $x = 3$.

   (e) For each critical point, determine whether $f(x)$ has a local maximum, local minimum, or neither, at that point. Justify your answer using either the First or Second Derivative Test. *(4 points)*

   **Answer.** Using the First Derivative Test:

   At $x = -3$, the derivative changes from negative to positive, so $f(x)$ has a **local minimum**.

   At $x = -2$, the derivative changes from positive to negative, so $f(x)$ has a **local maximum**.

   At $x = 0$, the derivative is negative on both sides of the critical point, so $f(x)$ **does not have a local extreme**.

   At $x = 3$, the derivative changes from negative to positive, so $f(x)$ has a **local minimum**.

2. At the annual basketball game between the Springfield University Atoms and the Shelbyville Technical Institute Tireburners, the Atoms' coach has used analytics to discover that the amount of time the cheerleaders spend singing their fight song, "Springfield Embiggens the Score," has a direct impact on the number of points their team scores.

   Their computer simulations have indicated that if the cheerleaders sing for $x$ minutes, then the Atoms will score

   $$A(x) = \frac{1}{3}x^3 - 3x^2 + 63 \quad \text{points;}$$

   the cheerleaders must sing for at least three minutes, and can sing for a maximum of 9 minutes. How long should they sing if they want to maximize the number of points scored by the Atoms?

   Your answer must be justified through the use of calculus. Graphing the function on your calculator and using the graph will receive no points. *(10 points)*

   **Answer.** Essentially, we are asked to find where the function $A(x)$ achieves its absolute maximum value on the interval $[3, 9]$. Because we have a continuous function on a finite closed interval, we just need to (i) find the critical points; (ii) evaluate at the critical points and at the endpoints; and (iii) compare the values obtained: the largest value we get gives the absolute maximum.

   To find the critical points, we find $A'(x)$. We have

   $$A'(x) = x^2 - 6x = x(x-6).$$

   This is always defined, and equal to 0 at $x = 0$ and $x = 6$; but $x = 0$ is not in the domain we are interested in, so the only critical point is $x = 6$.

   Evaluating, we have:

   $$A(6) = \frac{1}{3}(6^3) - 3(6^2) + 63 = 2(36) - 3(36) + 63 = 63 - 36 = 27.$$
   $$A(3) = \frac{1}{3}(3^3) - 3(3^2) + 63 = 9 - 27 + 63 = 72 - 27 = 45.$$
   $$A(9) = \frac{1}{3}(9^3) - 3(9^2) + 63 = 3(81) - 3(81) + 63 = 63.$$

   So the maximum value is 63, achieved at $x = 9$.

   This means that if the cheerleaders want to maximize the number of points scored by the Atoms, then they should sing for 9 minutes.

3. Use the first and second derivatives of

   $$f(x) = x^4 - 6x^2$$

   to find all intervals on which $f(x)$ is both **decreasing** and **concave up**, at the same time. *(10 points)*

   **Answer.** We have:

   $$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 4x(x + \sqrt{3})(x - \sqrt{3})$$
   $$f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x + 1)(x - 1).$$

   The critical points are $x = 0$, $x = -\sqrt{3}$, and $x = \sqrt{3}$. The points where $f''$ could change signs (points where $f(x)$ could change concavity) are $x = -1$ and $x = 1$. So we want to consider the

real line divided into six intervals: $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \sqrt{3})$, and $(\sqrt{3}, \infty)$. Doing a table, we have:

| | $(-\infty, -\sqrt{3})$ | $(-\sqrt{3}, -1)$ | $(-1, 0)$ | $(0, 1)$ | $(1, \sqrt{3})$ | $(\sqrt{3}, \infty)$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| $f''(x)$ | $+$ | $+$ | $-$ | $-$ | $+$ | $+$ |
| $f(x)$ | | | | | | |

So $f(x)$ is both decreasing and concave up on $(-\infty, -\sqrt{3})$ and on $(1, \sqrt{3})$.

4. We want to use a local linear approximation to estimate the value of

$$\sqrt[3]{65}.$$

(i) Write down the function $f(x)$ that you will use for the local linear approximation. *(1 point)*
**Answer.** While there are many possible answers, probably the simplest one is to take $f(x) = \sqrt[3]{x} = x^{1/3}$.

(ii) Write the value of $a$ (the "easy point nearby") that you will use for the approximation. *(1 point)*
**Answer.** Using $f(x) = x^{1/3}$, the "easy points nearby" is $a = 64$.

(iii) Find the values of $f(a)$ and $f'(a)$; give exact values, not approximations. *(2 points)*
**Answer.** We have $f(64) = \sqrt[3]{64} = 4$. For $f'(x)$, we have $f'(x) = \frac{1}{3}x^{-2/3}$, so

$$f'(64) = \frac{1}{3}(64)^{-2/3} = \frac{1}{3\left(\sqrt[3]{64}\right)^2} = \frac{1}{3(4)^2} = \frac{1}{3(16)} = \frac{1}{48}.$$

(iv) Use the local linear approximation determined by your answers in (i)–(iii) to give an approximation for $\sqrt[3]{65}$. Do not approximate this approximation: write the answer as an irreducible fraction. *(3 points)*
**Answer.** The general formula is $f(x) \approx f(a) + f'(a)(x - a)$ for $x$ near $a$. So

$$\sqrt[3]{65} = f(65) \approx 4 + \frac{1}{48}(65 - 64) = 4 + \frac{1}{48} = \frac{192 + 1}{48} = \frac{193}{48}.$$

That is, our approximation is $\sqrt[3]{65} \approx \frac{193}{48}$.

(v) Use $f''(x)$ to determine whether your approximation is an underestimate or an overestimate. *(1 point)*
**Answer.** Since $f''(x) = -\frac{2}{9}x^{5/3}$ and $f''(64) < 0$, that means that function is concave down, so the tangent lies above the graph. This means that the value above is an OVERESTIMATE.

5. Let $g(x)$ be the function

$$g(x) = x^3 - 3x^2.$$

(i) Find the critical points of $g(x)$. *(4 points)*
**Answer.** We have $g'(x) = 3x^2 - 6x = 3x(x - 2)$. So the critical points are $x = 0$ and $x = 2$.

(ii) For each critical point of $g(x)$, use the Second Derivative Test to determine if $g(x)$ has a local maximum or a local minimum at that point. If the Second Derivative Test is inconclusive, just say that it is inconclusive at that point. *(6 points)*
**Answer.** We have $g''(x) = 6x - 6$. Evaluating, we have:

$$g''(0) = 0 - 6 = -6,$$
$$g''(2) = 12 - 6 = 6.$$

Since $g''(0) < 0$ and $x = 0$ is a critical point, $g(x)$ has a **local maximum at $x = 0$.**
Since $g''(2) > 0$ and $x = 2$ is a critical point, $g(x)$ has a **local minimum at $x = 2$.**