Math 465 - Homework 1

Due Friday September 12

1. Give two reasons why the set of odd integers is not a group under addition.

2. Given real numbers a and b, with $a \neq 0$, let $T_{a,b} \colon \mathbb{R} \to \mathbb{R}$ be the function defined by
$$T_{a,b}(x) = ax + b.$$
   (a) Show that the composition of functions is a binary operation on the set
   $$S = \{T_{a,b} \mid a, b \in \mathbb{R}\}$$
   and express the function $T_{a,b} \circ T_{r,s}$ in the form $T_{\alpha,\beta}$ for suitable α and β.

   (b) Is $\circ$ an associative operation on S?

   (c) Are there values of a and b such that $T_{a,b} \circ T_{r,s} = T_{r,s} \circ T_{a,b} = T_{r,s}$ for all $r, s \in \mathbb{R}$? If so, what are they?

   (d) If the answer to (c) was "yes", then given fixed $r, s \in \mathbb{R}$, do there exist real numbers ρ and σ such that $T_{r,s} \circ T_{\rho,\sigma} = T_{\rho,\sigma} \circ T_{r,s} = T_{a,b}$, where a, b are the numbers you found in (c)? If so, express ρ and σ in terms of r and s.

   (e) Is S a group under $\circ$?

3. Show that $\{1, 2, 3\}$ is not a group under multiplication modulo 4, but that $\{1, 2, 3, 4\}$ is a group under multiplication modulo 5.

4. Show that $\{5, 15, 25, 35\}$ forms a group under multiplication modulo 40.

5. Show that the group $\mathsf{GL}(2, \mathbb{R})$ of invertible 2×2 matrices with real coefficients is non-Abelian by exhibiting a pair of matrices A and B for which $AB \neq BA$.