Math 465 - Homework 2

Due Friday September 19

1. Let G be a group with operation $\cdot$. We define THE OPPOSITE GROUP G^{op}, by taking the same underlying set as G, and defining the operation $\odot\colon G \times G \to G$ by
$$a \odot b = b \cdot a.$$
   (i) Prove that G^{op} is a group.
   (ii) Prove that $(G^{\text{op}})^{\text{op}} = G$.
2. Let G be a group. Prove that G is Abelian if and only if for every $a, b \in G$, we have $(ab)^{-1} = a^{-1}b^{-1}$.
3. Let G be a group, and $a, b \in G$. Prove that if $(ab)^2 = a^2b^2$, then $ab = ba$.
4. Let G be a group. Prove that if $g^2 = e$ for every $g \in G$, then G is Abelian (that is, for all $a, b \in G$, $ab = ba$).
5. Determine the order of each element of $U(14)$.
6. In the group $\mathbb{Z}_{12}$ of integers modulo 12 under addition modulo 12, find $|a|$, $|b|$, and $|a + b|$ in each of the following cases:
   (a) $a = 6$, $b = 2$;
   (b) $a = 3$, $b = 8$;
   (c) $a = 5$, $b = 4$.
7. Let G be a group, and let $a \in G$. Prove that $|a| = |a^{-1}|$, meaning that either they are both infinite, or they are both finite and equal to each other.
8. Let G be a group, and let $a, b \in G$. Prove that $|ab| = |ba|$, meaning that either they are both infinite, or they are both finite and equal to each other.
9. Let $G = D_4$ be the dihedral group of order 8.
   (a) Show that for every $g \in G$, we have $g^4 = R_0$ (the identity of G).
   (b) Show that for every $a, b \in G$, we have $(ab)^4 = a^4b^4$.
   (c) Show that G is not Abelian.