Math 465 - Homework 3

Due Friday September 26

1. Let G be a group, and let $a, x \in G$.

   (i) Prove that for every integer n,

   $$(axa^{-1})^n = ax^n a^{-1}.$$

   SUGGESTION: Use induction on n to prove it for $n \geq 0$, then take inverses to prove it for negative n.

   (ii) Let G be a group, $a, x \in G$. Prove that $|x| = |axa^{-1}|$. SUGGESTION: Show that the set of k for which $x^k = e$ is the same as the set of k for which $(axa^{-1})^k = e$.

2. If $n > 2$ and n is even, show that D_n has a subgroup of order 4.

3. Let G a group, and let H, K be subgroups. Show that if $hk = kh$ for every $h \in H$ and $k \in K$, then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G.

4. (i) Find all the generators of the groups $\mathbb{Z}_6$, $\mathbb{Z}_8$, and $\mathbb{Z}_{20}$.

   (ii) Let $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$ be cyclic groups of order 6, 8, and 20, respectively. Find all the generators of $\langle a \rangle$, of $\langle b \rangle$, and of $\langle c \rangle$.

5. Let G be a group, and let $a \in G$ be an element with $|a| = 15$. Compute the orders of each of the following elements of G:

   (i) a^3, a^6, a^9, and a^{12}.

   (ii) a^5 and a^{10}.

   (iii) a^2, a^4, a^8, and a^{14}.

6. In $\mathbb{Z}$, find all generators of the subgroup $\langle 3 \rangle$.

7. In $\mathbb{Z}$, find a generator for the subgroup $\langle 10 \rangle \cap \langle 12 \rangle$.

8. In $\mathbb{Z}$, show that $\langle n \rangle \subseteq \langle m \rangle$ if and only if m divides n.

9. In $\mathbb{Z}$, if $n, m \in \mathbb{Z}$, what is a generator for $\langle n \rangle \cap \langle m \rangle$?

10. Let G be an Abelian group, and let

    $$H = \big\{ g \in G \;\big|\; |g| \text{ divides } 12 \big\}.$$

    (i) Prove that H is a subgroup of G.

    (ii) Is there anything special about 12, or would your proof be valid if 12 were replaced by some other positive integer?

    (iii) State the general result derived from your answer of (ii).