Math 465 - Homework 1
SOLUTIONS
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1. Give two reasons why the set of odd integers is not a group under addition.

   Answer. Almost anything that could go wrong goes wrong here.

   For starters, addition of odd integers is not an operation: that is, if we let $\mathscr{O}$ be the set of odd integers, then addition is not a function $\mathscr{O} \times \mathscr{O} \to \mathscr{O}$, because the sum of two odd integers is never an odd integer.

   Also, the set does not have a neutral element relative to addition. And because it does not have a neutral element relative to addition, it also cannot have inverses relative to addition. □

2. Given real numbers a and b, with $a \neq 0$, let $T_{a,b}\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$T_{a,b}(x) = ax + b.$$

   (a) Show that the composition of functions is a binary operation on the set

$$S = \{T_{a,b} \mid a, b \in \mathbb{R}\}$$

   and express the function $T_{a,b} \circ T_{r,s}$ in the form $T_{\alpha,\beta}$ for suitable α and β.
   Proof. Indeed, a calculation gives

$$T_{a,b} \circ T_{r,s}(x) = T_{a,b}(rx + s) = a(rx + s) + b = (ar)x + (as + b) = T_{ar,as+b}(x).$$

   Thus, $T_{a,b} \circ T_{r,s} = T_{ar,as+b}$. And since $a \neq 0$ and $r \neq 0$, then $ar \neq 0$. So composition is a binary operation on the set S. □

   (b) Is $\circ$ an associative operation on S?
   Answer. We can see this in two ways. The quick and clever way is to note that this is just composition of functions, and we know that composition of functions is always associative.
   The second way is to just compare the result of computing $T_{a,b} \circ (T_{r,s} \circ T_{w,y})$ and computing $(T_{a,b} \circ T_{r,s}) \circ T_{w,y}$ using the formula above. We have:

$$\begin{aligned} T_{a,b} \circ (T_{r,s} \circ T_{w,y}) &= T_{a,b} \circ T_{rw,ry+s} = T_{a(rw),a(ry+s)+b};\\ (T_{a,b} \circ T_{r,s}) \circ T_{w,y} &= T_{ar,as+b} \circ T_{w,y} = T_{(ar)w,(ar)y+(as+b)}. \end{aligned}$$

   Since $a(rw) = (ar)w$, and $a(ry + s) + b = ary + as + b = (ar)y + (as + b)$, we conclude that $T_{a,b} \circ (T_{r,s} \circ T_{w,y}) = (T_{a,b} \circ T_{r,s}) \circ T_{w,y}$. □

   (c) Are there values of a and b such that $T_{a,b} \circ T_{r,s} = T_{r,s} \circ T_{a,b} = T_{r,s}$ for all $r, s \in \mathbb{R}$? If so, what are they?
   Answer. Yes: if we take $a = 1$ and $b = 0$, we have $T_{1,0}(x) = 1x + 0 = x$, so $T_{1,0} = \mathrm{Id}_{\mathbb{R}}$. Therefore, $T_{1,0} \circ T_{r,s} = T_{r,s} \circ T_{1,0} = T_{r,s}$ for all $r, s \in \mathbb{R}$. □

   (d) If the answer to (c) was "yes", then given fixed $r, s \in \mathbb{R}$, do there exist real numbers ρ and σ such that $T_{r,s} \circ T_{\rho,\sigma} = T_{\rho,\sigma} \circ T_{r,s} = T_{a,b}$, where a, b are the numbers you found in (c)? If so, express ρ and σ in terms of r and s.
   Answer. Again, yes. Since $T_{r,s} \circ T_{\rho,\sigma} = T_{r\rho,r\sigma+s}$, we want $r\rho = 1$ and $r\sigma + s = 0$. This means that we must have $\rho = \frac{1}{r}$, which we can do and makes sense because $r \neq 0$ is assumed; and $\sigma = -\frac{s}{r}$ (again, since $r \neq 0$, this can be done).
   We can verify that this works:

$$\begin{aligned} T_{r,s} \circ T_{\frac{1}{r},-\frac{s}{r}} &= T_{r(\frac{1}{r}),r(-\frac{s}{r})+s} = T_{1,-s+s} = T_{1,0};\\ T_{\frac{1}{r},-\frac{s}{r}} \circ T_{r,s} &= T_{(\frac{1}{r})r,(\frac{1}{r})s-\frac{s}{r}} = T_{1,\frac{s}{r}-\frac{s}{r}} = T_{1,0}. \qquad \square \end{aligned}$$

(e) Is S a group under $\circ$?

Answer. Yes; we have that $\circ$ is an associative operation on S (parts (a) and (b)). That the operation has a two-sided identity, namely $T_{1,0}$ (part (c)). And that each $T_{r,s}$ has an inverse (part (d)). Thus, we have a group. $\square$

3. Show that $\{1, 2, 3\}$ is not a group under multiplication modulo 4, but that $\{1, 2, 3, 4\}$ is a group under multiplication modulo 5.

Proof. The set $\{1, 2, 3\}$ is not closed under multiplication modulo 4. Although $1 \times 1 = 3$, $2 \times 1 = 2$, $2 \times 3 = 2$, and $3 \times 3 = 1$ hold modulo 4, we have $2 \times 2 = 0$, which is not in the set $\{1, 2, 3\}$; so this cannot be a group.

On the other hand, the set $\{1, 2, 3, 4\}$ is closed under multiplication modulo 5, has an identity element (namely 1), and inverses:

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Multiplication modulo 5 is associative, and from the table above we see that it has an identity element an inverses, so this is a group. $\square$

4. Show that $\{5, 15, 25, 35\}$ forms a group under multiplication modulo 40.

Proof. Multiplication modulo 40 is associative. This set is closed under the operation, as the table below shows:

$\times$	5	15	25	35
5	25	35	5	15
15	35	25	15	5
25	5	15	25	35
35	15	5	35	25

Notice that given that table, we have that the identity element is 25: indeed, $25x = x25 = x$ for all $x \in \{5, 15, 25, 35\}$. And once we have identified the identity, we see that inverses exist: each element is its own inverse. So this forms a group. $\square$

5. Show that the group $\mathsf{GL}(2, \mathbb{R})$ of invertible 2×2 matrices with real coefficients is non-Abelian by exhibiting a pair of matrices A and B for which $AB \neq BA$.

Proof. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that both matrices are invertible (their determinant is nonzero), so they are both elements of $\mathsf{GL}(2, \mathbb{R})$. Computing AB and BA, we have:

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},$$
$$BA = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}.$$

So AB and BA are not equal; that is, $AB \neq BA$. This shows that $\mathsf{GL}(2, \mathbb{R})$ is non-Abelian. $\square$