MATH 483 – Spring 2026

FINAL EXAM

SOLUTIONS

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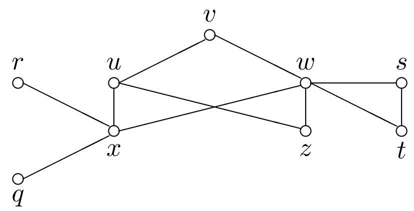
1. Let G be the following graph:



   (i) List all the bridges of G. *(2 points)*

   Answer. The bridges are the edges rx and qx. All other edges lie in cycles, so they are not bridges.

   (ii) List all the cut-vertices of G. *(2 points)*

   Answer. The cut vertices are x and w.

   (iii) List all the blocks of G *(4 points)*

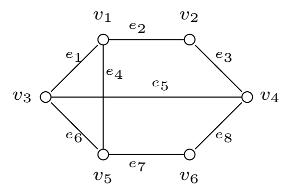
   Answer. The blocks (largest induced nonseparable subgraphs) are the subgraphs induced by the sets $\{r,x\}$, $\{q,x\}$, $\{u,v,w,x,z\}$, and $\{s,t,w\}$.

2. Let G be a connected regular graph that does not have an Eulerian circuit. Prove that if its complement $\overline{G}$ is connected, then $\overline{G}$ contains an Eulerian circuit. *(10 points)*

   Proof. Let n be the order of G, and suppose that G is r-regular. Because G does not have an Eulerian circuit, we know that r must be odd. We also know from class, as a consequence of the First Theorem of Graph Theory, that an r-regular graph of order n exists if and only if at least one of r and n are even; so we conclude that r is odd and n is even.

   Because G is r-regular, it follows that $\overline{G}$ is $(n-1-r)$-regular. And since n is even and r is odd, then $n-1-r$ is even. So if $\overline{G}$ is connected, then it is a connected graph in which every vertex has even degree, and therefore $\overline{G}$ contains an Eulerian circuit, as claimed.

3. Let G be the following graph:



   (i) Give the adjacency matrix for G. *(5 points)*

   Answer. This is a 6×6 matrix, which has (i,j) entry equal to 1 if v_i is adjacent to v_j, and 0 if v_i is not adjacent to v_j. We get:

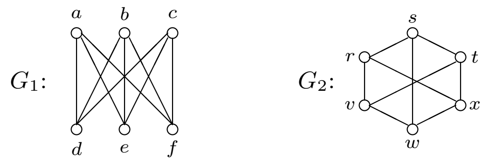
$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

   (ii) Give the incidence matrix for G. *(5 points)*

Answer. This is a 6×8 matrix; the (i, j) entry is 1 if vertex v_i is incident with edge e_j, and 0 otherwise. We have:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

4. Let G_1 and G_2 be the following two graphs:



Determine whether G_1 and G_2 are isomorphic or not.

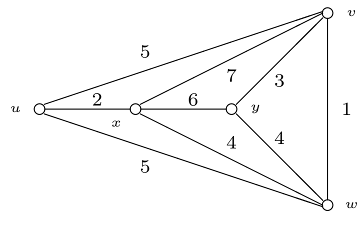
If you say they are isomorphic, give an isomorphism $\phi\colon V(G_1) \to V(G_2)$ by explicitly describing its value on the vertices $\{a, b, c, d, e, f\}$. If you say they are not isomorphic, then explain why the two graphs are not isomorphic. *(10 points)*

Answer. The graphs are isomorphic. To see that G_2 is also $K_{3,3}$, note that the vertices r, t, and w have no edges between them, but each of them is adjacent to all three remaining vertices s, v, and x; and that there are no edges between s, v, and x.

So any bijection that matches a, b, and c with either r, t, and w; or with s, v, and x, will do. For example:

$$\phi(a) = r \qquad \phi(d) = s$$
$$\phi(b) = t \qquad \phi(e) = v$$
$$\phi(c) = w \qquad \phi(f) = x.$$

5. Below is a weighted graph:

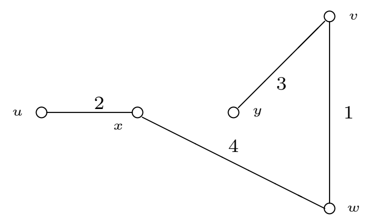


In the space below, give a minimum spanning tree for the graph, and state its total weight. *(10 points)*

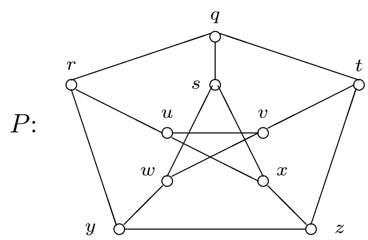
Answer. Either Kruskal's or Prim's algorithm will yield the same spanning tree.

For Kruskal's Algorithm, we start with vw; the next smallest weight edge is ux, then vy. Then we must add either wy or wx, but the first yields a cycle, so we add wx. At this point we have four edges in a graph of order 5, so we end. This gives the edges ux, xw, vw, and vy, of total weight $2 + 4 + 1 + 3 = 10$.

For Prim's Algorithm we start with an arbitrary vertex; say y, and add an edge of smallest weight incident on it; here yv. The next smallest weight edge that connects to this edge is vw. Then wx; and finally ux, leading to the same spanning tree:



6. Below is the Petersen graph:



   (i) Give a perfect matching for the Petersen graph. *(5 points)*

   Answer. There are many possibilities; one is to take the edges connecting the outer and inner cycles: qs, tv, xz, wy, and ru.

   (ii) Prove that the Petersen graph does not have two perfect matchings that are disjoint. *(7 points)*

   Proof. Let M be a perfect matching. If there were another perfect matching M' that is disjoint from M, then it would be a perfect matching for $P - M$, so let us consider $P - M$.

   Since P is 3-regular, then $P - M$ is 2-regular, and therefore each connected component is a cycle. We know that P is not Hamiltonian (proven in class), it cannot be a single cycle of length 10. And because the smallest cycle that is contained in P is of length 5, that is the smallest length that cycles in $P - M$ can have. So $P - M$ must be a union of two 5-cycles. But 5-cycles do not have perfect matchings. So there is no perfect matching M' of $P - M$. This shows that P cannot have two perfect matchings that are disjoint.

7. For the Petersen graph P of the previous problem, give an example of a vertex cut U for the graph that is not a minimum vertex cut, but such that no proper subset of U is a vertex cut. You do not need to prove that no proper subset of U is a vertex cut. *(10 points)*

   Proof. One example is $U = \{u, s, y, t\}$. This isolates the edge rq, so it is a vertex cut for the graph.

   Although you did not need to prove it, let me show that no proper subset of U disconnects P:

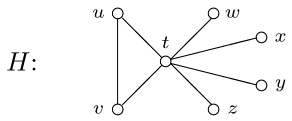
   The set $\{u, s, y\}$ does not disconnect P: we are left with vertices r, q, t, z, and x forming a path, and t, v, and w, also forming a path, with the two paths having t in common.

   The set $\{u, s, t\}$ does not disconnect P: we are left with the path q, r, y, z, and x; and the path y, w, and v.

   The set $\{u, y, t\}$ does not disconnect P: we are left with the path r, q, s, x, and z; and also s, w, and v.

   Finally, the set $\{s, y, t\}$ does not disconnect P: we are left with the path q, r, u, x, and z; and the path u, v, and w.

8. Let H be the graph below:



(i) Give a minimum vertex cover for H. *(3 points)*

Answer. A vertex cover will be incident on every edge; here we can take t, and then either u or v. So $\{t, u\}$ is a minimum vertex cover.

(ii) Give a maximum independent set of vertices for H. *(3 points)*

Answer. A maximum independent set of vertices is a set of vertices no two of which are adjacent. To achieve as large a number as possible we should omit t, take all of w, x, y, and z; and one of u or v. So for example, $\{u, w, x, y, z\}$.

Note that the size of a minimum vertex cover plus the size of a maximum independent set of vertices adds up to 7, as required by the formula $\alpha(G) + \beta(G) = n$.

(iii) Give a minimum edge cover for H. *(3 points)*

Answer. An edge cover for H is a set of edges that covers every vertex; here we need each of tw, tx, ty, and tz; and uv as well. So $\{uv, tw, tx, ty, tz\}$ is a minimum edge cover.

(iv) Give a maximum independent set of edges for H. *(3 points)*

Answer. We need edges that are not adjacent to each other. So we will take uv, and any one of edges adjacent to t *except* tu or tv. So for example, $\{uv, tw\}$.

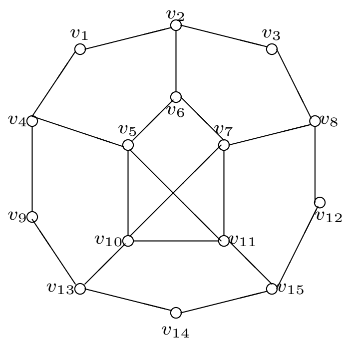
Note again that the sum of the sizes is 7, as required by the formula $\alpha'(G) + \beta'(G) = n$.

9. Let G be a 6-connected graph, and let u, v, w, and z be four distinct vertices of G, with u and v not adjacent. Prove that G contains two cycles C and C' that have only u and v in common, and neither of them contains w or z. *(10 points)*

   Proof. By Menger's Theorem, since we need at least 6 vertices to disconnect u from v, there are at least 6 internally disjoint u-v paths in G. At most one of them contains w, and at most one of them contains z. Let P_1, P_2, P_3, and P_4 be four internally disjoint u-v paths that do not contain either w or z.

   Let C consist of P_1 followed by P_2 in reverse order; this gives a cycle that includes u and v, but not w or z. Similarly, let C' consist of P_3 followed by P_4 in reverse order. Since P_1, P_2, P_3, and P_4 are pairwise internally disjoint, C and C' only have u and v in common.

10. Determine whether the graph below has a Hamiltonian cycle. If it does, give the cycle by listing the vertices in the order of the cycle. If it does not, explain how you know it does not. *(8 points)*



Answer. This graph is not Hamiltonian.

If we had a Hamiltonian cycle C, then because v_1, v_3, v_9, v_{12}, and v_{14} all have degree 2, the cycle C would need to include the two edges incident in each of those vertices. But that gives all the edges in the outer cycle

$$(v_1, v_2, v_3, v_8, v_{12}, v_{15}, v_{14}, v_{13}, v_9, v_4, v_1, v_2)$$

However, a cycle cannot include a smaller cycle, so there can be no Hamiltonian cycle in this graph.

NOTE: As an aside, note that the graph **does** have a Hamiltonian *path*: for example,

$$v_1, v_2, v_6, v_7, v_{11}, v_{10}, v_5, v_4, v_9, v_{13}, v_{14}, v_{15}, v_{12}, v_8, v_3.$$