Math 483 - Spring 26
Homework 1
SOLUTIONS
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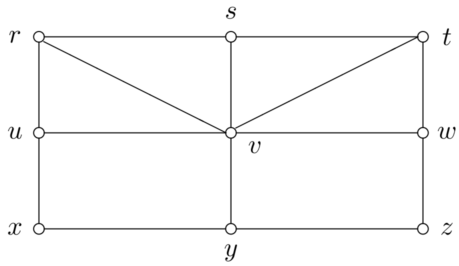
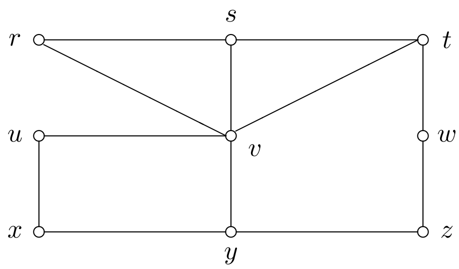
1. Let G be the graph below:

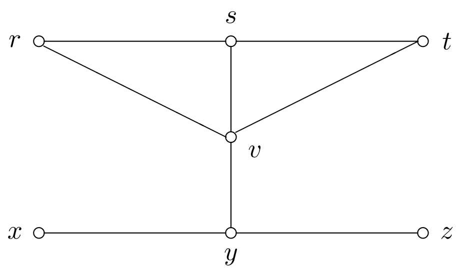


Let e and f be the edges $e = ru$ and $f = vw$. Draw the subgraphs $G - \{e, f\}$ and $G - \{u, w\}$.

Answer. Here is $G = \{e, f\}$:



Here is $G - \{u, w\}$:



2. For the graph above, give an example of each of the following, or explain why no such example exists:

(i) An x-y walk of length 6.

Answer. One possible example is $W = (x, u, r, s, t, v, y)$.

(ii) A v-w trail that is not a v-w path.

Answer. One possibility is $T = (v, s, t, v, w)$.

(iii) An r-z path of length 2.

Answer. There is no r-z path of length 2. Note that vertices w and y are the only vertices that are distance 1 from z; and r is not adjacent to either w or y. So the distance from r to z has to be at least three, and there can be no path of length 2.

(iv) An x-z path of length 3.

Answer. There is no x-z path of length 3. To have an x-z path of length 3, we would need to have an path of length 2 from x to either y or to w; but no such paths exist.

(v) An x-t path of length $d(x,t)$.

Answer. Note that $d(x,t) = 3$. The vertices that are distance 1 from x are u and y. The vertices that are distance 2 are then r, v, and z. So the distance to t is at least 3. And since $W = (x, y, v, t)$ is an x-t path of length 3, we have that the distance is at most 3. Thus, the distance is 3, and W is an example of a path as desired.

(vi) A circuit of length 10.

Answer. One example could be $C = (x, u, r, v, s, t, v, w, z, y, x)$. Note that this is not a cycle, since the vertex v is repeated.

(vii) A cycle of length 8.

Answer. One possible example is $C = (x, u, r, s, t, w, z, y, x)$.

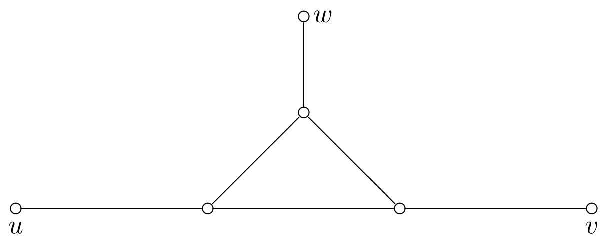
(viii) A geodesic of length $\operatorname{diam}(G)$.

Answer. Note that the furthest apart that two vertices of G are from each other is 3: indeed, we can get from v to every vertex except x and z in one step; and from v to x and z in two steps. So we can get from x to any vertex other than z in at most three steps by going to v first; and we can get from x to z in two steps. Symmetrically, the distance form z to any vertex is at most three. And the distance between any two vertices other than x and z is at most two, again by going through v if necessary. And since we already know that the distance from x to t is 3, we have that the diameter of G is exactly 3.

So what we need is a geodesic of length 3. One possibility is $P = (x, y, v, t)$.

3. Give an example of a connected graph G that includes three pairwise distinct vertices u, v, and w (and possibly other vertices), such that $d(u,v) = d(u,w) = d(v,w) = \operatorname{diam}(G) = 3$.

Answer. Here is an example:

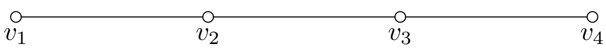


4. We saw in class that a graph G of order at least 3 is connected if and only if G contains two distinct vertices u and v such that both $G - u$ and $G - v$ are connected. One might guess the following statement would also be true:

Every connected graph G of order at least 4 contains three pairwise distinct vertices u, v, and w, such that each of $G - u$, $G - v$, and $G - w$ are connected.

However, this statement is false. Draw a counterexample; that is, a graph G with at least 4 vertices, that is connected, and show that no choice of three distinct vertices will saisfy the condition.

Answer. One example of such a graph would be a path of length 3:



Any choice of three pairwise distinct vertices must include either v_2 or v_3, and both $G - v_2$ and $G - v_3$ are disconnected. Yet G has order 4 and is connected.

5. Let G be a connected graph, and suppose that u and v are two distinct vertices in G. What is the minimum order of a connected subgraph of G that contains both u and v? Explain your answer.

   Answer. The minimum order is $d(u, v) + 1$, with a geodesic from u to v being the subgraph. Indeed, H is a connected subgraph that contains both u and v, then any u-v path in H is also a u-v path in G, and so contains at least $d(u, v)$ edges, and hence $d(u, v) + 1$ vertices. So the order of H is at least $d(u, v) + 1$. On the other hand, if $W = (u = v_0, v_1, \ldots, v_k = v)$ is a u-v geodesic, so $k = d(u, v)$, then $G[\{v_0, \ldots, v_k\}]$ is a connected graph that contains exactly $k + 1$ vertices, including u and v, so the minimum order is at most $d(u, v) + 1$. This proves the equality.