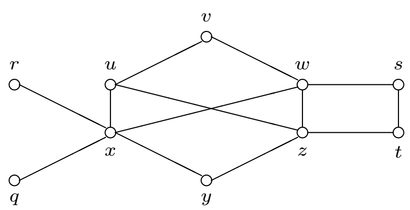
Math 483 - Spring 26
Homework 2
Solutions
Prof. Arturo Magidin

1. Let G be the following graph:



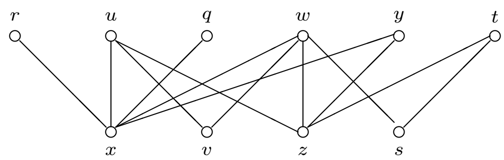
Determine whether G is bipartite or not. If it is bipartite, redraw it with vertices in two rows and all edges between the rows, clearly labeling the vertices. If it is not bipartite, explain why it is not bipartite.

Answer. This graph is bipartite, and we can verify this by noting that there are no odd cycles in the graph. We can then select a vertex, say r, and divide the graph into vertices that are an odd distance from r, and vertices that are an even distance from r.

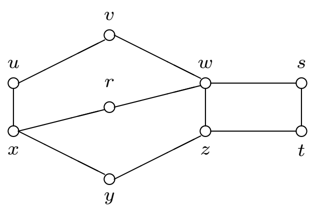
The vertices that are an odd distance from r are x (distance 1), v (distance 3), z (also distance 3), and s (distance 5).

The vertices that are an even distance from r are r (distance 0), u (distance 2), q (distance 2), w (distance 2), y (distance 2), and t (distance 4).

We get the following redrawing of G:



2. Let G be the following graph:



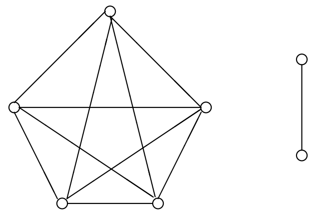
Determine whether G is bipartite or not. If it is bipartite, redraw it with vertices in two rows and all edges between the rows, clearly labeling the vertices. If it is not bipartite, explain why it is not bipartite.

Answer. This graph is not bipartite. Note that, for example, $C = (x, u, v, w, r, x)$ is a 5-cycle. Since the graph contains odd cycles, it is not bipartite.

3. Let $G = K_5$ and $H = K_2$.

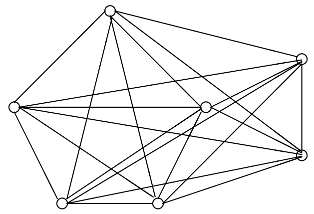
   (i) Draw $G \cup H$.

   Answer. This is just G alongside H, with no edges joining them.



(ii) Draw $G + H$.

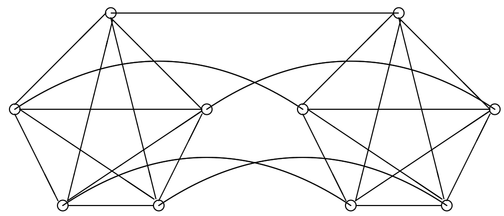
Answer. This is G alongside H, with edges joining each vertex of G to all vertices of H:



(iii) Draw $G \times H$.

Answer. We will one copy of G for for each vertex of H; then in the copies we join corresponding vertices in two copies if and only if the vertices of H that index the copies of G are joined in H.

Here we will have two copies of G, and we join corresponding vertices (since the two vertices of H are adjacent). Thinking of H horizontally instead of vertically, we have:



4. Give an example of the following, or explain why no example exists.

(i) A graph of order 7 with vertices of degree 1, 1, 1, 2, 2, 3, and 3.

Answer. There can be no example satisfying these conditions: recall that the sum of all degrees of a graph must equal twice the number of edges, and so in particular must be an even number. But here, we have $1 + 1 + 1 + 2 + 2 + 3 + 3 = 13$.

(ii) A graph of order 7 with vertices of degree 1, 2, 2, 2, 3, 3, and 7.

Answer. Again, there is no such example: the order of a graph is the number of vertices in the graph; and in a graph of order n, the degree of a vertex is at most $n - 1$. For a graph of order 7, the largest possible degree is 6 (if the vertex is adjacent to every other vertex). Here, we are being asked for a graph of order 7 with a vertex of degree 7, which is impossible.

5. The degree of each vertex of a graph of order 12 and size 31 is either 4 or 6. How many vertices of degree 4 does the graph have?

Answer. Let x be the number of vertices of degree 4, and y the number of vertices of degree 6. Since graph has order 12, we have that $x + y = 12$. Since the sum of the degrees is twice the size, we also have $4x + 6y = 62$. Thus, we are trying to solve the system of equations

$$\begin{aligned} x + y &= 12 \\ 4x + 6y &= 62. \end{aligned}$$

Substituting $y = 12 - x$ into the second equation, we obtain

$$4x + 6(12 - x) = 62,$$

which yield $-2x + 72 = 62$, or $10 = 2x$. Thus, $x = 5$. That is, there must be 5 vertices of degree 4 (and therefore, 7 vertices of degree 6).

6. Suppose that G is a disconnected graph that has exactly two odd vertices. Prove that the two odd vertices must be in the same connected component of G.

   Proof. Notice that a connected component of a graph G is itself a graph, in which the degrees of the vertices agree with the degrees in G, because every edge incident to the vertex in G is also incident in the component. That means that if H is a connected component of G, then

$$\sum_{v \in V(H)} \deg(v) = \text{twice the size of the component.}$$

   Suppose then that the two odd vertices u and v are in different connected components of G, say H_u and H_v. Then the sum of the degree of vertices in H_u is an odd number, since all vertices but u are even vertices (none of them are equal to v), and u is an odd vertex. As this is impossible, we reach a contradiction. The contradiction arises from the undischarged assumption that the two odd vertices are in different connected components, so we conclude that it must be the case that they are in the same connected component of G, as claimed.

7. A graph G has the property that every edge of G joins an odd vertex with an even vertex. Prove that G is bipartite and has even size.

   Proof. Let U be the set of odd vertices, and W the set of even vertices; as every vertex is either even or odd, but not both, this is a partition of $V(G)$ into two sets. By assumption, every edge of G connects a vertex of U with a vertex of W, so we see that G is bipartite.

   We proved in class that if G is bipartite, with parts U and W, then the sum of the degrees of the vertices in U equals the sum of the degrees of the vertices in W, and this equals the size of G. Thus,

$$\text{size}(G) = \sum_{w \in W} \deg(w),$$

   and since $\deg(w)$ is even for each $w \in W$, we see that the size of G is a sum of even numbers, and therefore is itself even, ad claimed.

8. Let G be a connected graph that has the property that for every two distinct vertices u and v, either all u-v paths have odd length, or else all u-v paths have even length. Prove that G is bipartite.

   Proof. We show that G does not contain any odd cycles.

   Let $C = (x_0, x_1, \ldots, x_n = x_0)$ be a cycle in G of length n, with $n \geq 3$. We want to show that n is even.

   We have two x_0-x_1 paths by going around the cycle in different directions: $P_1 = (x_0, x_1)$, of length 1; and $P_2 = (x_0 = x_n, x_{n-1}, \ldots, x_1)$, of length $n - 1$. Since any two paths between x_0 and x_1 are either both of even length or both of odd length, and P_1 is of odd length, it follows that P_2 is also of odd length. Thus, $n - 1$ is odd, and therefore n is even, which is what we wanted to prove.

   Since every cycle in G is even, it follows by the Theorem proven in class that G is bipartite, as claimed.