Math 483 - Spring 26
HOMEWORK 3
Solutions
Prof. Arturo Magidin

1. Let $r \geq 1$. Show that if G is both r-regular and bipartite with partite sets U and W, then $|U| = |W|$ (that is, U and W have the same number of vertices).

   Proof. Because G is bipartite, we know that the size of G is the sum of the degrees of the vertices in either of the parts. In particular, the sums are equal to each other. That is, we have

$$\sum_{u\in U} \deg(u) = \sum_{w\in W} \deg(w).$$

   Because we are also assuming that the graph is r-regular, then all degrees equal r. So we have:

$$r|U| = \sum_{u\in U} \deg(u) = \sum_{w\in W} \deg(w) = r|W|.$$

   Cancelling r we obtain $|U| = |W|$, which is what we wanted to show.

2. Let G be a graph of order n, and let $\overline{G}$ be its complement.

   (a) Prove that $\delta(G) + \delta(\overline{G}) \leq n - 1$.

   Proof. Because $uv \in E(G)$ if and only if $uv \notin E(\overline{G})$, we have that two vertices of G will have an edge between them in *either* G *or* in $\overline{G}$, but not both. Therefore, for each vertex u,

$$\deg_G(u) + \deg_{\overline{G}}(u) = n - 1.$$

   So a vertex with the smallest possible degree in $\overline{G}$ has *largest* possible degree in G; in other words, $\deg_{\overline{G}}(u) = \delta(\overline{G})$ if and only if $\deg_G(u) = \Delta(G)$. Therefore,

$$\delta(\overline{G}) = (n-1) - \Delta(G).$$

   Therefore,

$$\begin{aligned}\delta(G) + \delta(\overline{G}) &= \delta(G) + (n-1) - \Delta(G)\\ &= (n-1) + (\delta(G) - \Delta(G))\\ &\leq n-1,\end{aligned}$$

   with the last inequality because $\delta(G) \leq \Delta(G)$, and therefore, $\delta(G) - \Delta(G) \leq 0$.

   (b) Show that G is r-regular if and only if $\overline{G}$ is $(n-1-r)$-regular.

   Proof. We have:

$$\begin{aligned}G\text{ is }r\text{-regular} &\iff \text{For every vertex } v,\ \deg_G(v) = r\\ &\iff \text{For every vertex } v,\ \deg_{\overline{G}}(v) = (n-1) - r\\ &\iff \overline{G}\text{ is }(n-1-r)\text{-regular.}\end{aligned}$$

   (c) Prove that $\delta(G) + \delta(\overline{G}) = n - 1$ if and only if G is regular.

   Proof. From the inequality in part (a) we have that:

$$\begin{aligned}\delta(G) + \delta(\overline{G}) = n-1 &\iff \delta(G) - \Delta(G) = 0\\ &\iff \delta(G) = \Delta(G).\end{aligned}$$

   But the smallest possible degree of a vertex equals the largest possible degree of a vertex if and only if all vertices have the same degree; that is, if and only if G is regular, as claimed.

(d) Prove that G is regular if and only if there is a vertex v of G such that $\deg_G(v) = \delta(G)$ and $\deg_{\overline{G}}(v) = \delta(\overline{G})$.

Proof. As we noted above, for any vertex v we have $\deg_{\overline{G}}(v) = (n-1) - \deg_G(v)$; and $\deg_{\overline{G}}(v) = \delta(\overline{G})$ if and only if $\deg_G(v) = \Delta(G)$.

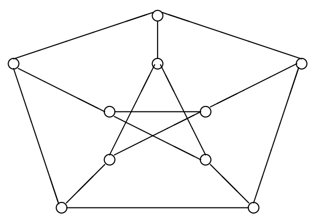
If G is regular, then so is $\overline{G}$, so every vertex has $\deg_G(v) = \delta(G)$ and $\deg_{\overline{G}}(v) = \delta(\overline{G})$.

Conversely, if there is a vertex v with $\deg_G(v) = \delta(G)$ and $\deg_{\overline{G}}(v) = \delta(\overline{G})$, then the second equality yields $\deg_G(v) = \Delta(G)$; so we conclude that $\delta(G) = \deg_G(v) = \Delta(G)$, so G is regular.

3. A *spanning subgraph* of a graph G is a subgraph H of G with $V(H) = V(G)$ (that is, H is obtained by removing edges from G, but keeping all the vertices).

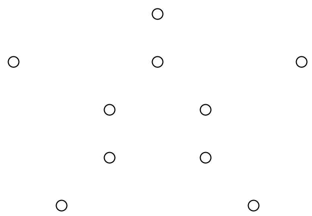
(a) Find spanning subgraphs G_r of the Petersen graph such that G_r is r-regular, for $r = 0$, 1, 2, and 3.

Answer. For reference, recall that the Petersen graph is:



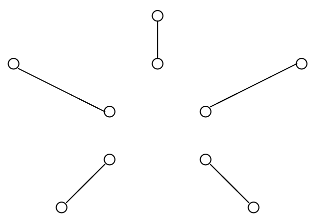
There is only one 0-regular spanning subgraph:

G_0:



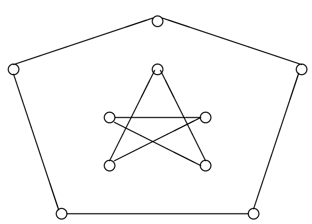
For G_1, we have multiple options. Here is one:

G_1:



Likewise, there are many options for G_2; here is one:

G_2:



The Petersen graph itself is 3-regular, so that is the only possibility for G_3.

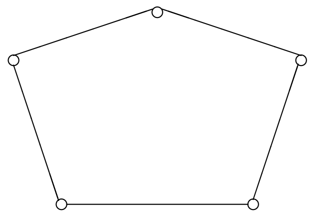
(b) Find induced subgraphs F_r of the Petersen graph such that F_r is r-regular, for $r = 0$, 1, 2, and 3.

Answer. Note that in (a) we were looking for spanning subgraphs, here we are looking for *induced* subgraphs. There are many possibilities for $r = 0$, but the simplest is to take a single vertex. But we could take larger graphs.

For $r = 1$, we can take just two connected vertices and the edge between them, or two pairs of connected vertices that are not connected to each other.

For $r = 2$, we can take the outside cycle or the inside one:

F_2:



And for F_3 we can take the whole Petersen graph.

4. If the sequence $x, 7, 7, 5, 5, 4, 3, 2$ is graphical, what are the possible values of x, with $0 \leq x \leq 7$? Note that we need not have the resulting sequence be non-increasing.

Answer. Note that because there are 8 numbers, we know that $x \leq 7$. Because there are at least two 7s, we also know that $x \geq 2$. And because there is a 2, we know that x cannot be 7, because then there would be three vertices that are adjacent to every other vertices, so the smallest possible number would be 3. Thus, we have, as a first restriction, that $2 \leq x \leq 6$.

Since there are already five odd numbers, and we know that there must be an even number of vertices with odd degree, we also know that x must be odd. So we must have either $x = 3$ or $x = 5$.

To verify that both of these values are valid, we can use the Havel-Hakimi Theorem.

Case 1: $x = 5$. Applying the Havel-Hakimi Theorem to 7, 7, 5, 5, 5, 4, 3, 2, we see that this sequence is graphical if and only if

$$6, 4, 4, 4, 3, 2, 1$$

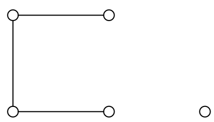
is graphical, if and only if

$$3, 3, 3, 2, 1, 0$$

is graphical, if and only if

$$2, 2, 1, 1, 0$$

is graphical. And this is indeed graphical, as witnessed for example by:



Case 2: $x = 3$. Applying the Havel-Hakimi theorem to 7, 7, 5, 5, 4, 3, 3, 2, we have that the sequence is graphical if and only if

$$6, 4, 4, 3, 2, 2, 1$$

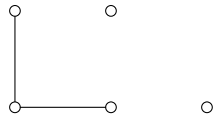
is graphical, if and only if

$$3, 3, 2, 1, 1, 0$$

is graphical, if and only if

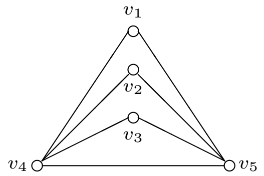
$$2, 1, 0, 1, 0$$

is graphical; equivalently, if and only if $2, 1, 1, 0, 0$ is graphical, which again is indeed the case, as witnessed by



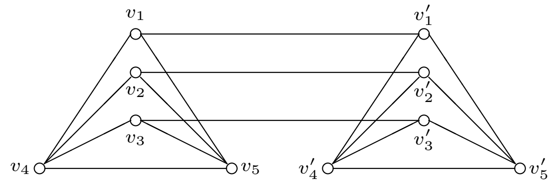
So we conclude that we must have either $x = 5$ or $x = 3$, and either of those values is possible.

5. Let G be the following graph:

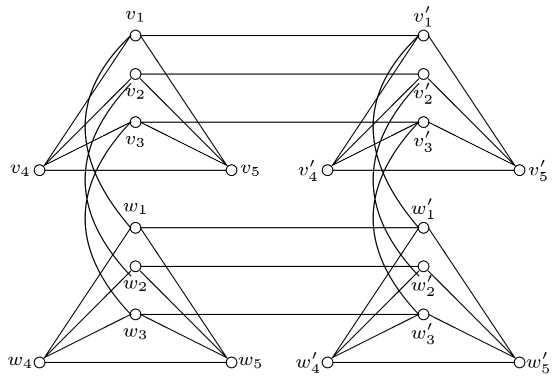


Find a 4-regular graph that contains G as an induced subgraph.

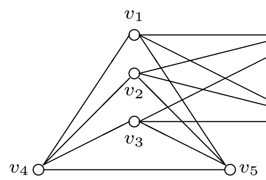
Answer. One possibility is to use the construction given in the proof of König's Theorem. We copy G and join the corresponding vertices of degree smaller than 4:



And then do it again with this new graph:



A much smaller solution is to add two vertices that connect to each of v_1, v_2, and v_3 (to make the latter three have degree four each); then join the two new vertices together to make them also have degree 4. That is:



6. Let G be the graph from Problem 5.

(a) Compute the adjacency matrix A of G.

Answer. The adjacency matrix is:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

(b) Use the adjacency matrix to determine the number of length 3 walks from v_1 to v_5.

Answer. We compute A^3, and look at the $(1,5)$ entry. We have:

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 4 & 3 \\ 1 & 1 & 1 & 3 & 4 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 4 & 3 \\ 1 & 1 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 & 7 & 7 \\ 2 & 2 & 2 & 7 & 7 \\ 2 & 2 & 2 & 7 & 7 \\ 7 & 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 7 & 6 \end{pmatrix}.$$

So there are seven walks of length 3 from v_1 to v_5.

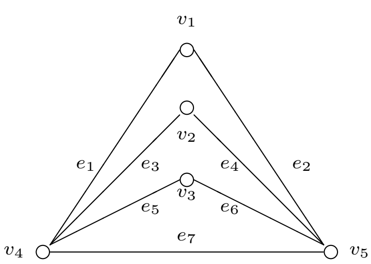
(c) List all v_1-v_4 walks of length 3.

Answer. From A^3 above we see that there are also a total of seven walks of length 3 from v_1 to v_4. We have:

(i) (v_1, v_4, v_1, v_4);
(ii) (v_1, v_4, v_2, v_4);
(iii) (v_1, v_4, v_3, v_4);
(iv) (v_1, v_4, v_5, v_4);
(v) (v_1, v_5, v_1, v_4);
(vi) (v_1, v_5, v_2, v_4);
(vii) (v_1, v_5, v_3, v_4).

7. Let G be the graph from Problem 5. Label the edges and compute the incidence matrix of G.

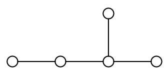
Answer. I'll draw the graph slightly larger so that the edge labels are not terribly crowded. We label the edges:



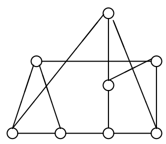
So the incidence matrix is a 5×7 matrix given by:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

8. Find a 3-regular graph of minimum order that contains the following graph H as an induced subgraph:



Answer. We need to add at least two new vertices so that we can add two edges to the vertices of degree 1; since we need an even number of vertices and we currently have five vertices, we in fact need to add at least three new vertices. This will suffice: we can make each of those adjacent to two of the degree 1 vertices; the make one of them also adjacent to the degree 2 vertex, giving both degree 3; and finally make the other two new vertices adjacent to each other:



As we noted above, we must add at least three new vertices, so this graph is a 3-regular graph of minimum order containing H as an inducedd subgraph.

9. Use the Havel-Hakimi Theorem to determine if the given sequences are graphical or not:

   (a) s_1: $5, 3, 3, 3, 3, 2, 2, 2, 1$.
   Answer. Applying the Theorem once, we obtain the sequence

   $$2, 2, 2, 2, 1, 2, 2, 1$$

   or equivalently, 2, 2, 2, 2, 2, 2, 1, 1. This sequence is graphical, because we can take a path of length 7:

   So the sequence s_1 is graphical as well.

   (b) s_2: $6, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1$.
   Answer. Applying the Theorem, we obtain the sequence

   $$2, 2, 2, 2, 1, 1, 2, 2, 1, 1$$

   Again, this sequence is graphical: we can realize it with two disconnected paths whose lengths add up to 8. They will have four vertices of degree 1 (the four endpoints), and six "midpoints" total. So the sequence s_2 is also graphical.

   (c) s_3: $6, 5, 5, 4, 3, 2, 1$.
   Answer. Here, the Theorem yields the sequence

   $$4, 4, 3, 2, 1, 0$$

This sequence is not graphical: note that would be only five vertices that have edges incident in them, which means that the two vertices of degree 4 would have to be adjacent to every vertex that is not isolated. But that would mean that the smallest positive degree that occurs is 2. As this sequence is not graphical, neither is s_3.

If it was not clear that the above sequence was not graphical, we can apply the Theorem again to obtain

$$3, 2, 1, 0, 0.$$

Again, this sequence is not graphical, because there are only three vertices of positive degree, so the largest possible degree would be 2, not 3.

Again, if it was not clear whether the above sequence was graphical or not, we can apply the theorem again, which would yield

$$1, 0, -1, 0$$

which is clearly not graphical, since it contains negative numbers.