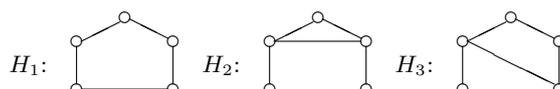**Math 483 - Spring 26**
HOMEWORK 4
Solutions
Prof. Arturo Magidin

1. Give an example of three non-isomorphic graphs of order 5 and size 5.

   Answer. There are, of course, many, many possible answers. Here is one:

   Consider the following three graphs:

   H_1: H_2: H_3:

   How can we easily tell these graphs are pairwise non-isomorphic? Note that in H_1 every vertex has degree 2; in H_2 there are exactly two vertices of degree 1; and in H_3 there is exactly one vertex of order 1. Since isomorphic graphs have the same degree sequences (up to reordering), none these three graphs can be isomorphic to another one of these three graphs.

2. Below are four graphs. Which pairs of graphs are isomorphic, and which pairs are not? Justify your answer.

   G_1: G_2: G_3: G_4:
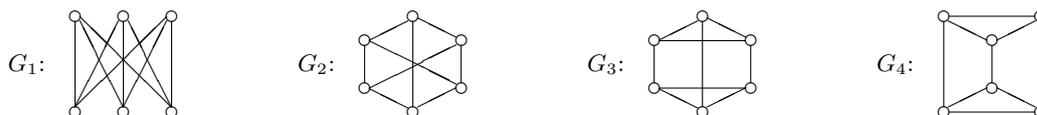
   Answer. G_1 is bipartite, and both G_3 and G_4 have cycles of odd length, so we know that G_1 is not isomorphic to either G_3 or G_4. On the other hand, G_1 *is* isomorphic to G_2: label the vertices of G_2, starting at the top and going clockwise, as a, b, c, d, e, and f. Note that each of a, c, and e are adjacent to each of b, d, and f, and not to each other; and none of b, d, or f are adjacent to each other. So G_2 is also the bipartite graph $K_{3,3}$. In particular, it is not isomorphic to G_3 or to G_4.

   And G_3 is isomorphic to G_4; the simplest way to see this is to imagine taking the vertices at the top and bottom of G_3, and shifting them towards each other until they are past the horizontal edges, so that graph G_3 becomes G_4. This tells us how to define the isomorphisms.

   In summary, G_1 and G_2 are isomorphic to each other; G_3 and G_4 are isomorphic to each other; and neither of G_1 nor G_2 is isomorphic to either G_3 or G_4.

3. Let G_1 and G_2 be two graphs, with vertex sets $V(G_1) = \{u_1, v_1, w_1, x_1, y_1, z_1\}$ and $V(G_2) = \{u_2, v_2, w_2, x_2, y_2, z_2\}$. If v_1 has degree 3 and is adjacent to a vertex of degree 2, while v_2 has degree 3 and is *not* adjacent to any vertex of degree 2, can we conclude that $G_1 \not\cong G_2$? Explain your answer.

   Answer. No, this is not enough to conclude that $G_1 \not\cong G_2$. While we know that if there is an isomorphism between G_1 and G_2, then v_1 will be mapped to a vertex of degree 3 that is adjacent to a vertex of degree 2, that vertex need not be v_2: it could be some *other* vertex of degree 3.

   For example, consider the following two labeled graphs:

$$G_1 \qquad\qquad G_2$$
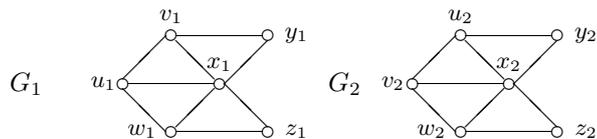
These two graphs are isomorphic. Note that v_1 has degree 3, and is adjacent to a vertex of degree 2 (namely, y_1). Meanwhile, v_2 has degree 3, but is not adjacent to any vertex of degree 2.

(Here, an isomorphism could be defined by mapping u_1 to v_2, v_1 to u_2, and the remaining four vertices to their correspondingly labeled vertices; the issue is that there are multiple vertices of degree 3, so the fact that one of them cannot be the image of v_1 does not, by itself, preclude the existence of an isomorphism mapping v_1 to one of the *others*.)

4. Can a disconnected graph be self-complementary? Explain.

   Answer. No, a disconnected graph cannot be self-complementary. Recall that we proved in class that if G is disconnected, then its complement $\overline{G}$ is connected. Since a connected graph cannot be isomorphic to a disconnected graph, it follows that a disconnected graph G cannot be isomorphic to $\overline{G}$, and hence cannot be self-complementary.

5. Consider the (unlabeled) graphs G, H_1, H_2, and H_3 below:
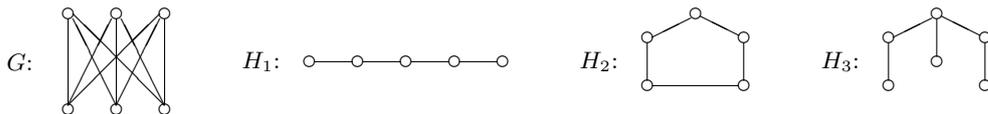
   Determine which H_i are isomorphic to a subgraph of G. Explain your answer.

   Answer. So I can describe the embeddings, take the following labeling of G:
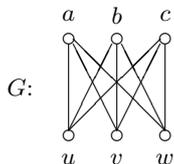
   We can find a path of length 5 in G, by taking for example (a, u, b, v, c); we can embed H_1 as that path, so H_1 is isomorphic to a subgraph of G. Note that although H_1 is not isomorphic to an *induced* subgraph of G, the problem only asks whether it is isomorphic to a subgraph of G.

   Since G is bipartite, it does not contain any odd cycle, so it cannot contain any subgraph isomorphic to H_2.

   Finally, H_3 is isomorphic to a subgraph of G: labeling the vertices of H_3 so they correspond to vertices of G, we could have for example:
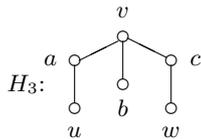
   This shows H_3 is isomorphic to a subgraph of G; but note again that it is not isomorphic to an *induced* subgraph of G.

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6. Suppose we have a collection $G_1, \ldots, G_n$ of graphs, some pairs of which are isomorphic and some pairs of which are not. Show that there is an even number of graphs that are isomorphic to an odd number of graphs. HINT: Create a graph of order n in which v_i and v_j, $i \neq j$, are adjacent if and only if G_i is isomorphic to G_j.

Proof. As the hint suggests, consider a graph with vertices $v_1, \ldots, v_n$, where two distinct vertices v_i and v_j are adjacent if and only if G_i is isomorphic to G_j.

The degree of v_i in this graph is the number of graphs, other than itself, to which the graph is isomorphic. So a graph is isomorphic to an odd number of other graphs if and only if the corresponding vertex is odd. Since we know that in any graph there is an even number of odd vertices, it follows that there is an even number of graphs G_i that are isomorphic to an odd number of other graphs on the list.

7. Does there exist a trio of graphs, G_1, G_2, and G_3, with exactly two pairs of the graphs isomorphic to each other, but the third pair not isomorphic to each other?

Answer. The answer is "no". For suppose that G_1 and G_2 are isomorphic, and also G_2 and G_3 are isomorphic. In that case, it must be the case that G_1 and G_3 are *also* isomorphic, because "is isomorphic to" is a transitive relation.

8. Prove or disprove: if G and H are two connected graphs of order n, and there exists a bijection $\phi \colon V(G) \to V(H)$ with the property that for all $u, v \in V(G)$, $d_G(u, v) = d_H(\phi(u), \phi(v))$ (that is, ϕ preserves distances), then $G \cong H$.

Answer. The statement is true. Suppose that we have a bijection ϕ with the given property. We claim that ϕ is in fact an isomorphism. To prove that, we must show that for any two vertices u and v of G, u is adjacent to v if and only if $\phi(u)$ is adjacent to $\phi(v)$.
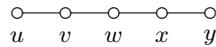
Note that two vertices are adjacent if and only if the distance between them is 1. Thus, we have

$$
\begin{aligned}
uv \in E(G) &\iff d_G(u, v) = 1 \\
&\iff d_H(\phi(u), \phi(v)) = 1 \\
&\iff \phi(u)\phi(v) \in E(H).
\end{aligned}
$$

Thus, ϕ is an isomorphism between G and H, proving that $G \cong H$.

9. Determine all automorphisms of a path of length 5.

Answer. Consider a path of length 5:

There are only two vertices of degree 1. So any automorphism must send either u to itself, or to y.

If u is sent to itself, then v must be sent to a vertex that is adjcent to u, which must be v; then w must be sent to a vertex of degree 2 that is adjacent to v, which must be itself; and x must be sent to a vertex of degree 2 that is adjacent to w and is not v (since v is already in the image), hence x is mapped to itself, and that leaves y to be mapped to itself. That is, you get the identity map.

If u is sent to y, then v must be setn to the vertex adjacent to y, hence to x; then w must be mapped to itself, x to v, and y to u. So in this case we obtain the map that "flips" the path.

These are the only two possibilities.

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10. Determine all automorphisms of a cycle of length 3.

   Answer. Consider a cycle of length 3:
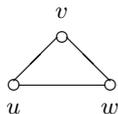
We can map u to any of the three vertices. Once we decide where to map u, then we can map v to any of the remaining two vertices. And once the images of u and v have been chosen, the image of w is forced. So in total we obtain six automorphisms:

$$\phi_1 : u \longmapsto u \qquad\qquad \phi_2 : u \longmapsto u$$
$$v \longmapsto v \qquad\qquad\qquad v \longmapsto w$$
$$w \longmapsto w \qquad\qquad\qquad w \longmapsto v$$

$$\phi_3 : u \longmapsto v \qquad\qquad \phi_4 : u \longmapsto v$$
$$v \longmapsto w \qquad\qquad\qquad v \longmapsto u$$
$$w \longmapsto u \qquad\qquad\qquad w \longmapsto w$$

$$\phi_5 : u \longmapsto w \qquad\qquad \phi_6 : u \longmapsto w$$
$$v \longmapsto u \qquad\qquad\qquad v \longmapsto v$$
$$w \longmapsto v \qquad\qquad\qquad w \longmapsto u$$

   (For those who know some Group Theory, this is the dihedral group of order 6, or equivalently, the permutation group S_3.)