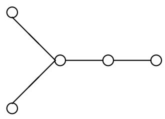
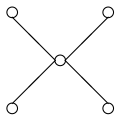
Math 483 - Spring 26
Homework 5
Solutions
Prof. Arturo Magidin

1. (a) Draw all trees of order 5.

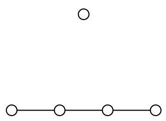
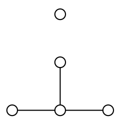
   Answer. A tree of order 5 can have 2, 3, or 4 ends. The first possibility is a path of length 4; the last is a star. And the case of 3 ends also corresponds to a unique tree of order 5. They are:

   (b) Draw all forests of order 5 that are not trees.

   Answer. There are three forests with two components; two forests with three components; one forest with four components; and one forest with five components.

   The three forests with 2 components are: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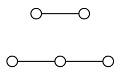
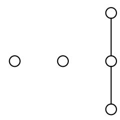


   The two forests with 3 components are: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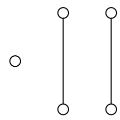
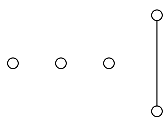


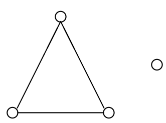
   The forest with 4 components is:



   Finally, the forest with 5 components is:

2. Give an example of a graph of order n and size $n-1$ that is not a tree.

   Answer. Such a graph cannot be connected (since a connected graph of order n and size $n-1$ must be a tree), and must contain a cycle (since a forest of order n that is not connected has at most $n-2$ edges). Since it must contain a cycle, one component must have at least three vertices. And so the smallest example has $n=4$; that is, four vertices and three edges:



3. Show that G is a graph and $\delta(G) \geq 2$, then G contains a cycle.

   Proof. Because $\delta(G) \geq 2$, any connected component of G has at least three vertices.

   If G_1 is a connected component of G, and G_1 has no cycles, then it is a tree; and if G_1 is a tree, then it has at least two end vertices. As end vertices have degree 1, it follows that $\delta(G_1) = 1$. Thus, if G_1 contains no cycles, then $1 = \delta(G_1) \geq \delta(G)$.

   By contrapositive, if $\delta(G) \geq 2$, then G_1 must contain cycles, and thus so does G.

4. A tree T has 50 end-vertices, an equal number of vertices of degree 2, degree 3, degree 4, and degree 5, and no vertex of degree greater than 5. What is the order of T?

   Answer. Let x be the number of vertices of degree 2, and thus also the number of vertices of degrees 3, 4, and 5. The total number of vertices is then $n = 50 + 4x$.

   On the other hand, twice the size is the sum of the degrees; since the size of a tree of order n is $n - 1$, this means that $2(n - 1) = 50 + (2 + 3 + 4 + 5)x = 50 + 14x$.

   So we have the following two equations:

$$n = 50 + 4x$$
$$2n - 2 = 50 + 14x$$

   Substituting the value of n into the second equation, we have

$$100 + 8x - 2 = 50 + 14x,$$

   which gives $48 = 6x$, or $x = 8$. Therefore, $n = 50 + 4x = 82$. So T has order 82.

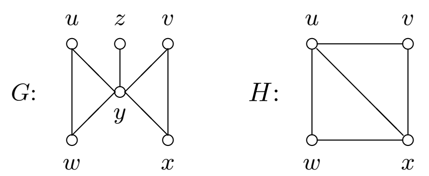
5. Determine all regular trees (that is, all graphs that are both regular and trees).

   Answer. From Problem 2, we know that an r-regular graph with $r \geq 2$ must have cycles, and therefore not be a tree. So if a tree is regular, then it is either 0-regular or 1-regular.

   Because trees must be connected, a 0-regular tree consists of a single vertex (and no edges), and a 1-regular tree consists of two vertices and an edge between them.

   And these are all the regular trees.

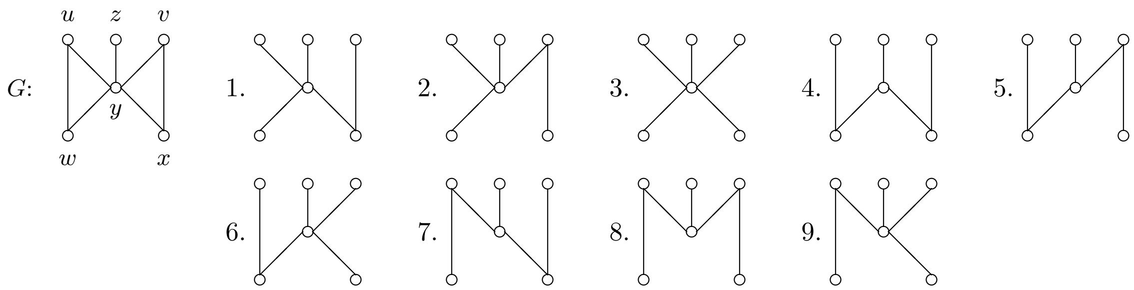
6. Let G and H be the following graphs:



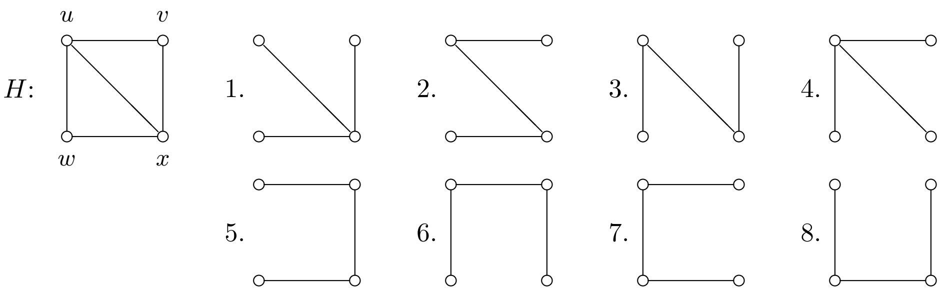
   (a) Determine all spanning trees for G and all spanning trees for H.

   Answer. Every spanning tree for G must contain the edge yz. It must omit one edge from the uwy triangle (three possibilities), and one from the vxy triangle (three possibilities). This gives nine spanning trees for G.

   We have:



For H, if the spanning tree includes ux, then it must exclude one of the edges in the triangle uwx, and one of the edges from the triangle uvx, giving four such spanning trees. If it omits ux, then it must also omit one edge from the square $uvxw$, giving four possibilities; for a total of eight spanning trees.



(b) For each, determine which spanning trees are isomorphic to each other.

Answer. In the case of G, the spanning trees numbered as 1, 2, 6 and 9 are all isomorphic to each other and no other spanning tree. Spanning trees 4, 5, 7, and 8 are isomorphic to each other and not to any other spanning tree. Finally, spanning tree 3 is a star, and is not isomorphic to any other of the spanning trees.

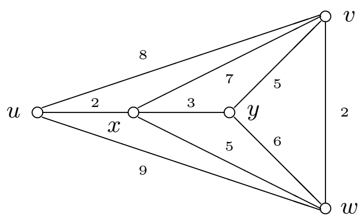
In the case of H, spanning trees 2, 3, 5, 6, 7, and 8 are all paths of length 3; and spanning trees 1 and 4 are isomorphic to each other and not to any other spanning tree.

7. Show that an edge e of a connected graph G is a bridge if and only if e belongs to every spanning tree of G.

Proof. If e is a bridge, then any spanning subgraph of G that does not contain e is not connected, hence not a tree. Thus, if e is a bridge and T is a spanning tree of G, then e belongs to T.

Conversely, suppose that e is not a bridge. Then $G - e$ is connected, and so has a spanning tree; a spanning tree of $G - e$ is also a spanning tree of G, and of course cannot contain e. So if e is not a bridge, then there is at least one spanning tree of G to which e does not belong. By contrapositive, if e belongs to every spanning tree of G, then it is a bridge.

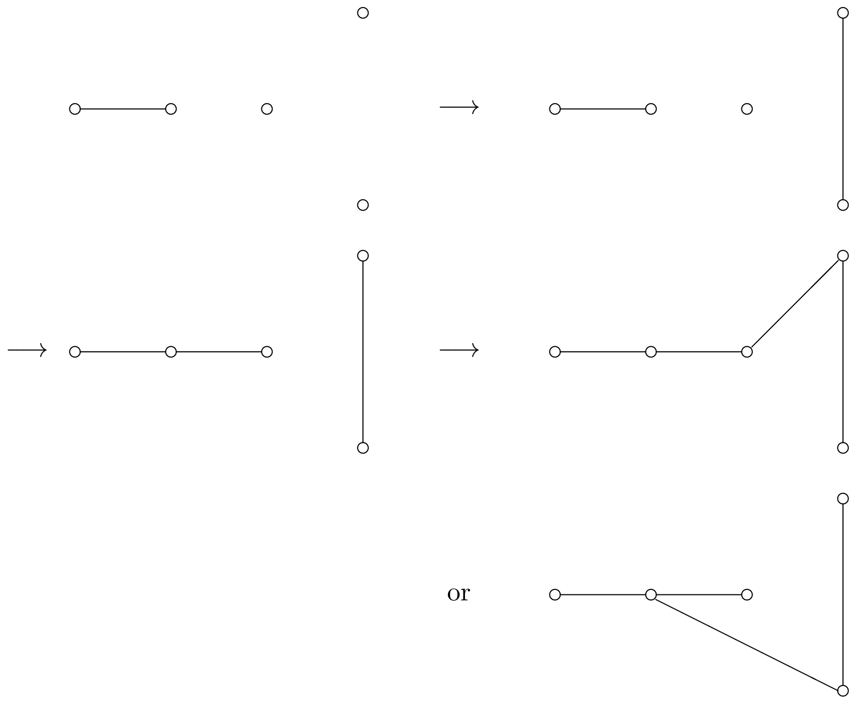
8. Apply both Kruskal's and Prim's Algorithms to find a minimum spanning tree in the weighted graph pictured below. In each case, show the steps you are following to arrive at the minimum spanning tree.



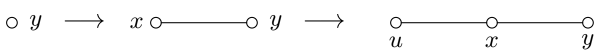
Answer. Both algorithms take four steps (since the graph has five vertices).

In the case of Kruskal's algorithm, there are two possible starts depending on which edge of weight 2 we select first, but in either case we will select the other second. The third step will select the edge xy of weight 3; then we have a choice of edges of degree 5 (either xw or yv), and that will end the process.

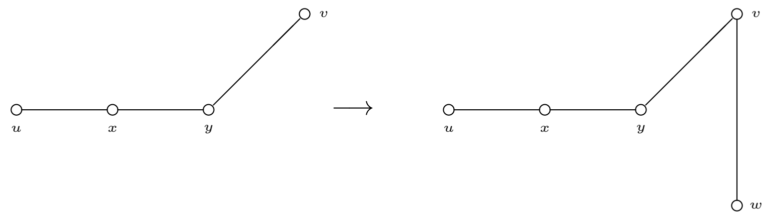
So we have:



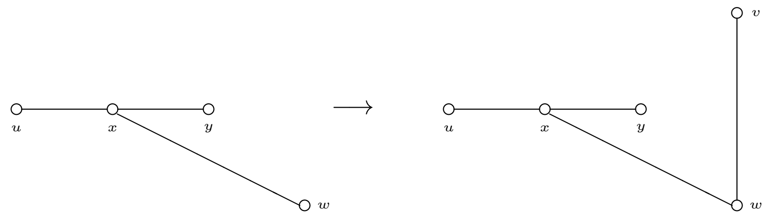
In the case of Prim's algorithm, it depends on which vertex we start with. For example, starting with y, we will first add the edge xy. Then the edge ux. Then we have a choice of adding yv or xw; and finally we will add vw. So we have:



and then either



or else



Other starts are possible, but in any case we will end up with one of these two spanning trees, which have weight 12.