Math 483 - Spring 26
HOMEWORK 6
Solutions
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1. Show that if a tree T has a vertex v with $\deg(v) > 2$, then T has more than two end-vertices.

   Proof. Recall that a tree is connected, so every vertex has degree at least 1; that trees always have at least two ends; and that a tree of order n has size $n-1$. The sum of the degrees must therefore equal $2(n-1) = 2n-2$.

   Let $v_1, \ldots, v_n$ be the vertices of T, with v_1 and v_2 the two end vertices that we are guaranteed to have. Then, looking at the sum of the degrees of the remaining $n-2$ vertices, we must have

   $$\sum_{j=3}^{n} \deg_T(v_j) = 2(n-2).$$

   Since there are $n-2$ summands, if $\deg_T(v_j) > 2$ then there must be $\deg_T(v_j) - 2$ vertices of degree 1 for the equality to hold. Thus, any vertex in T with degree greater than 2 forces at least one additional end-vertex in addition to v_1 and v_2.

2. Suppose G is a connected graph of order n, and v is a cut-vertex of G. What is the largest possible number of connected components that $G-v$ may have? Explain how to draw a graph that achieves the upper bound you give.

   Answer. The largest possible number of connected components is $n-1$, which is achieved with a star of order n: the cut vertex is the center vertex, and removing it leaves $n-1$ isolated vertices.

3. Let G be a connected graph. Prove that if a vertex v is an end-vertex of a spanning tree of G, then v is not a cut-vertex of G.

   Proof. If v is an end-vertex of a tree T, then $T-v$ is connected, and hence a tree. Since $T-v$ is a subgraph of $G-v$ which is both a tree and contains all vertices of $G-v$, it follows that $T-v$ is a spanning tree of $G-v$. The existence of a spanning tree implies that $G-v$ must be connected.

   Since $G-v$ is connected, we conclude that v is not a cut-vertex of G.

4. Let G be a connected graph with at least two vertices, and let v be a vertex of G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

   Proof. Assign weights to every edge of G by letting edges that are incident with v have weight 1, and every other edge have weight 2. If we follow Prim's Algorithm to construct a spanning tree of minimum weight starting with the vertex v, the next $\deg_G(v)$ steps will add one of the edges incident in v to the minimum spanning tree, because they are all of least degree, connecting a new vertex to the ones already chosen, and without creating a cycle because the graph that consists of v, all edges incident in v, and the other end vertices is a star. Only after we add all these edges will the algorithm move on to edges not incident on v.

   As we already know, Prim's Algorithm will create a spanning tree of minimum weight; in particular, a spanning tree. Thus, we see that there must exist a spanning tree that contains all edges incident with v, as claimed.

5. Prove that if G is a connected graph that has exactly two vertices that are not cut-vertices, then G is a path.

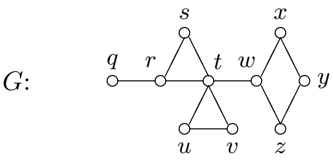
   Proof. Let v_1 and v_2 be the two vertices that are not cut-vertices. From Problem 3 we know that any vertex which is an end-vertex of a spanning tree for G is not cut-vertex. Thus, in any spanning tree, only v_1 and v_2 are end-vertices.

From Problem 1 we know that if a spanning tree for G has a vertex with degree greater than 2, then T has more than two end-vertices. So in any spanning tree for G, only v_1 and v_2 are end-vertices, and all other vertices have degree exactly 2.

And from Problem 4 we know that if v is a vertex of G, then there is a spanning tree of G that includes all edges that are incident in v; so in G, both v_1 and v_2 must have degree 1 and all other vertices have degree exactly 2.

This means that G is a path: it is connected, there are two vertices of degree 1, and every other vertex has degree 2.

6. For the graph G below, determine the cut-vertices, bridge, and blocks of G:



Answer.

(i) **Cut vertices:** The cut-vertices are r, t, and w. Indeed, the edge qr is a bridge with r of degree 3, so r is a cut-vertex; and tw is a bridge, with both t and w of degree greater than 1, so they are both cut-vertices. No other vertex is a cut vertex: removing q, s, u, v, x, or z clearly do not disconnect G.

(ii) **Bridges:** The only edges that are not in any cycle are qr and tw; these are the only bridges.

(iii) **Blocks:** Bridges form their own blocks. So one block is the edge qr together with vertices q and r; and another block consists of the edge tw together with vertices t and w.

The other three blocks are the individual cycles: the cycle $wxyz$, the cycle rst, and the cycle tuv. This accounts for all edges, so these are all the blocks in G.

7. Let G be a connected graph, v a cut-vertex of G, and let G_1 be a connected component of $G - v$. Show that the induced subgraph of G determined by $V(G_1) \cup \{v\}$ (that is, the vertices of G_1 plus the vertex v) is connected.

Proof. For simplicity, let us denote by H the induced subgraph of G with vertex set $V(G_1) \cup \{v\}$.

Let $x, y \in V(H)$ be two vertices. If $x, y \in V(G_1)$, then there is an x-y path in G_1, and hence in H.

Suppose then that one of x and y is in $V(G_1)$, and the other is equal to v.

Because v is a cut vertex, there exists a vertex $w \in V(G)$ that is not in G_1 and does not equal v (namely, pick any other connected component of $G - v$ and a vertex w in that component). And since G is connected, there is an x-w path P in G. Because this path does not exist in $G - v$, it follows that P goes through v; and since P starts in G_1, the first vertex in P that is not in G_1 must be equal to v. That is, we can write

$$P = (x, u_1, \ldots, u_k, v, u_{k+2}, \ldots, u_{r-1}, w),$$

with $u_1, \ldots, u_k \in V(G_1)$ Therefore, $(x_,u_1, \ldots, u_k, v)$ is an x-v path in H.

Thus, there is a path between any two vertices in $V(H)$, proving that H is connected.

8. Show that if v is a cut-vertex of G, then v is not a cut-vertex of the complement $\overline{G}$ of G.

Proof. Note that $\overline{G} - v = \overline{G - v}$: indeed, they have the same vertices (namely $V(G) - \{v\}$); and for any $x, y \in V(G)$,

$$\begin{aligned} xy \in E(\overline{G} - v) &\text{ if and only if } xy \in E(\overline{G}) \text{ and } x \neq v, y \neq v \\ &\text{ if and only if } xy \notin E(G) \text{ and } x \neq v, y \neq v \\ &\text{ if and only if } xy \notin E(G - v) \text{ and } x \neq v, y \neq v. \end{aligned}$$

We know that because v is a cut-vertex, then $G - v$ is disconnected. We also know that the complement of a disconnected graph is connected, so $\overline{G - v} = \overline{G} - v$ is connected. This tells us that v is not a cut-vertex of $\overline{G}$, which is what we wanted to prove.