Math 483 - Spring 26
HOMEWORK 7
Solutions
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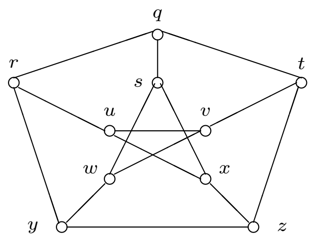
1. Consider the Petersen Graph:



   (i) Give an example of a minimum vertex cut for the Petersen Graph.

   Answer. Because the Petersen Graph is 3-regular, we know that a minimum vertex cut will have at most 3 vertices. Because any two vertices lie in a common cycle, we also know that the graph is non-separable, so a vertex cut has at least 2 vertices.

   In fact, there is no pair of vertices that will disconnect the graph. If we remove two vertices from the outer 5-cycle (consisting of r, q, t, z, and y), since every vertex is connected to the inner 5-cycle (which consists of s, u, v, w, and x), the graph would remain connected. Likewise, if we remove two vertices from the inner 5-cycle then the outer vertices are connected and the remaining inner vertices connect to them. And if we remove one vertex from the outer cycle and one from the inner cycle, then the remaining 4 vertices in each cycle remain connected, and there remain edges between the two cycles. Thus, a minimum vertex cut will necessarily have three vertices.

   At this point, we can obtain a minimum vertex cut by taking any vertex, say q, and removing the three vertices that it is adjacent to; in this case r, s, and t.

   (ii) Give an example of a vertex cut U for the Petersen Graph that is not a minimum vertex cut, and such that no proper subset of U is a vertex cut for the Petersen Graph.

   Answer. Consider $U = \{r, t, w, x\}$. I claim that U is a vertex cut for the Petersen Graph.

   Indeed, in $G - U$ consists of three connected components: the one consisting of q and s; the one consisting of u and v; and the one consisting of y and z.

   However, no proper subset of U is a vertex cut. By symmetry, it suffices to consider $\{r, w, x\}$ and $\{r, t, w\}$.

   If we remove the vertices in $\{r, w, x\}$, then the vertices q, t, z, and y remain connected; s is connected to q; and u, v, and w remain connected to t. Thus, $G - \{r, w, x\}$ is connected.

   If we remove the vertices in $\{r, t, w\}$, then the vertices s, u, v, and x remain connected; q is connected to s, and y and z are connected to x. Thus, $G - \{r, t, w\}$ is connected.

   So U is a vertex cut that is not a minimum vertex cut (it has one too many elements), but no proper subset of U is a vertex cut for the Petersen Graph.

2. Give an example of a 2-connected graph that is not 3-connected.

   Answer. To be 2-connected we must have $\kappa(G) \geq 2$; to be 3-connected we must have $\kappa(G) \geq 3$. So we are looking for a graph with $\kappa(G) = 2$. An example would be a cycle C_n with $n \geq 4$: it is nonseparable, since any two vertices lie in a cycle; thus $\kappa(C_n) \geq 2$. And if we remove two non-adjacent vertices then it disconnects the graph, so $\kappa(C_n) \leq 2$, giving the equality.

3. Give an example of a 2-edge connected graph that is not 3-edge connected.

   Answeer. A cycle C_n with $n \geq 3$ satisfies $\lambda(C_n) = 2$, as removing any two edges will disconnect the graph. That is an example of a 2-edge connected graph that is not 3-edge connected.

4. Prove that if G is k-connected, and v is a vertex of G, then $G - v$ is $(k-1)$-connected.

   Proof. Let U be a subset of $V(G - v)$ that has fewer than $k - 1$ vertices. Then $U \cup \{v\}$ is a subset of $V(G)$ that has fewer than k vertices. Because G is assumed to be k-connected, we know that $G - (U \cup \{v\})$ is connected and nontrivial.

   Because $(G - v) - U = G - (U \cup \{v\})$, that means that any vertex cut for $G - v$ must have at least $k - 1$ vertices, proving that $\kappa(G - v) \geq k - 1$, as desired.

5. Prove that if G is ℓ-edge connected and e is an edge of G, then $G - e$ is $(\ell - 1)$-edge connected.

   Proof. The argument is similar to the one above: let X be a subset of $E(G - e)$ that has fewer than $\ell - 1$ edges. Then $X \cup \{e\}$ has fewer than ℓ elements, so $G - (X \cup \{e\})$ is connected, given that G is ℓ-edge connected. Since $(G - e) - X = G - (X \cup \{e\})$, it follows that no set with fewer than $\ell - 1$ edges of $G - e$ will be an edge-cut for $G - e$, and hence that $\lambda(G - e) \geq \ell - 1$, as claimed.

6. Let G be a graph of order n, and let v be a vertex of G. Prove that if $\deg_G(v) = n - 1$ and U is a vertex cut for G, then $v \in U$.

   Proof. Because G has order n, and $\deg_G(v) = n - 1$, it follows that v is adjacent to every other vertex in G.

   We claim that if U is a subset of $V(G)$ that does **not** contain v, then $G - U$ is connected (that is, U is not a vertex cut for G), and therefore that every vertex cut for G must include v. Indeed, let $x, y \in V(G) - U$, $x \neq y$. If one of them equals v, then it is adjacent to the other because v is adjacent to every vertex in G. If neither one is equal to v, then we know that each is adjacent to v, and so (x, v, y) is an x-y path in $G - U$. In either case, x and y are connected, proving that U is not a vertex cut of G.

   By contrapositive, if U is a vertex cut for G, then $u \in U$.

7. Let G be a 5-connected graph and let u, v, and w be three distinct vertices of G. Prove that G contains two cycles, C and C', that have only u and v in common, and neither of them contains w.

   Proof. Because G is 5-connected, we know that for any pair of vertices in G there are at least 5 internally disjoint paths between them.

   So there are at least 5 internally disjoint u-v paths in G. Because they are internally disjoint, at most one of them includes w. That means that we have (at least) four u-v paths that are internally disjoint and do not include w, say

$$\begin{aligned}
P_1 &= (u = u_{10}, u_{11}, \ldots, u_{1n_1} = v), \\
P_2 &= (u = u_{20}, u_{21}, \ldots, u_{2n_2} = v), \\
P_3 &= (u = u_{30}, u_{31}, \ldots, u_{3n_3} = v), \\
P_4 &= (u = u_{40}, u_{41}, \ldots, u_{4n_4} = v).
\end{aligned}$$

   All the vertices u_{ij}, $j \neq 0$, $j \neq n_i$ are pairwise distinct, and different from u, v, and w.

   Then we have the following two cycles that include both u and v, have no other vertex in common, and do not contain w:

$$\begin{aligned}
C_1 &= (u = u_{10}, u_{11}, \ldots, u_{1n_1} = v = u_{2n_2}, u_{2(n_2-1)}, \ldots, u_{21}, u_{20)} = u) \\
C_2 &= (u = u_{30}, u_{31}, \ldots, u_{3n_3} = v = u_{4n_4}, u_{4(n_4-1)}, \ldots, u_{41}, u_{40)} = u).
\end{aligned}$$

8. Suppose that G is a k-connected graph, and u, $v_1, \ldots, v_k$ are $k+1$ distinct vertices of G. Show that we can find path P_1 from u to v_1, P_2 from u to v_2, ..., and P_k from u to v_k such that $P_1, \ldots, P_k$ are internally disjoint.

   Proof. Let H be a graph obtained by adding a new vertex w to G, adding edges from v_i to w for $i = 1, \ldots, k$.

   We proved in class that because G is k-connected, then H is also k-connected.

   Now consider the vertices u and w in H; these are two distinct vertices in a k-connected graph, so as a consequence of Menger's Theorem we know that there are at least k pairwise internally disjoint u-w paths $Q_1, \ldots, Q_k$ in H. However, since w is only adjacent to $v_1, \ldots, v_k$, it follows that the paths must go through $v_1, \ldots, v_k$. Reordering if necessary, we may assume that Q_i goes through v_i. Let P_i be the path Q_i with its final vertex w removed.

   Since P_i does not include w, each of $P_1, \ldots, P_k$ are paths in G. Also, P_1 is a path from u to v_1; P_2 is a path from u to v_2, ..., and P_k is a path from u to v_k. And these paths are internally disjoint (in fact, the only vertex they have in common is u).