Math 483 - Spring 26

HOMEWORK 8

Solutions

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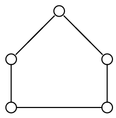
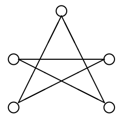
1. For each of the following, give an example of a graph G satisfying the following conditions (here, $\overline{G}$ represents the complement of G):

   (i) A graph G such that both G and $\overline{G}$ have Eulerian circuits.

   Answer. Let G be a cycle of length 5. This has an Eulerian circuit. The complement is also a cycle of length 5:

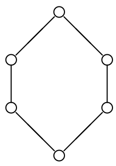
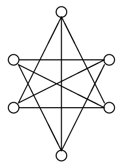
   G:  $\overline{G}$: 

   Since both G and $\overline{G}$ are cycles, they both have Eulerian circuits.

   (ii) A graph G that has an Eulerian circuit, but $\overline{G}$ does not.

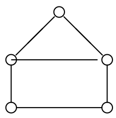
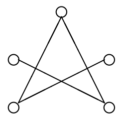
   Answer. An easy answer is to take a complete graph of odd order (so every vertex has even degree), as the complement will be disconnected.

   But we can also find an example in which both G and $\overline{G}$ are connected and satisfy the given conditions. For instance, let G be a cycle of length 6, which has an Eulerian circuit. The complement is a 3-regular graph with six vertices, so it cannot have an Eulerian circuit.

   G:  $\overline{G}$: 

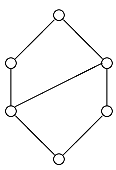
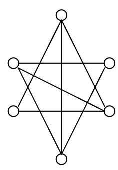
   (iii) A graph G such that neither G nor $\overline{G}$ contain an Eulerian circuit, and both G and $\overline{G}$ do contain an Eulerian trail.

   Answer. Here is an example, also of order 5:

   G:  $\overline{G}$: 

   (iv) A graph G such that neither G nor $\overline{G}$ contain an Eulerian circuit, G *does* contain an Eulerian trail, and $\overline{G}$ does not contain an Eulerian trail.

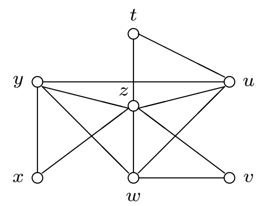
   Answer. Again, let us give an example in which $\overline{G}$ is connected, but does not have an Eulerian trail. Here is an example of order 6: note that because G has four vertices of degree 2 and two of degree 3 (hence has an Eulerian trail but no circuit), $\overline{G}$ has four vertices of degree 3 and two of degree 2, so it does not have an Eulerian circuit nor an Eulerian trail:

   G:  $\overline{G}$: 

2. Show that if G is connected and has order at least 2, then there is a closed walk in G that includes every vertex at least once, and every edge exactly twice.

   Proof. Construct a multigraph H that has the same vertex set as G, but where each edge has been replaced with two parallel edges (two edged connecting the same two vertices). In this (multi)graph, every vertex has even order, so the graph has an Eulerian circuit. This circuit describes a closed walk in G in which every vertex is visited (because G is connected), and every edge is traversed exactly twice, once for each of the two parallel edges in H.

3. Let G be the graph below:



   Determine whether G is Hamiltonian or not; if it is Hamiltonian, explicitly write out a Hammiltonian cycle. If it is not, prove that no Hamiltonian cycle exists.

   Answer. The graph G does not contain a Hamiltonian cycle. If it had one, then because t has degree two, both tz nad tu would need to be in the cycle. By the same token, so do xy and xz; and also vw and vz. But this tells us that z is incident with three edges n the cycle: tz, vz, and xz. This is impossible, since vertices in a Hamiltonian cycle are incident to only two edges in the cycle.

4. Recall that if G and H are graphs, then $G + H$ is the obtained by taking a copy of G, a copy of H, and then adding an edge between each vertex of G and every vertex of H.

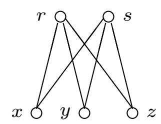
   Suppose that G is a 3-regular graph of order 12, and H is a 4-regular graph of order 11. Is $G + H$ Eulerian?

   Answer. In general, for graphs G and H of order m and n respectively, we have that if $x \in V(G)$, then $\deg_{G+H}(x) = \deg_G(x) + n$; and if $y \in V(H)$, then $\deg_{G+H}(y) = \deg_H(y) + m$.

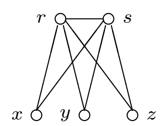
   A vertex x in $V(G)$ will therefore have $\deg_{G+H}(x) = 3 + 11 = 14$; and a vertex y in $V(H)$ will have $\deg_{G+H}(y) = 4 + 12 = 16$. So every vertex will have even degree, and therefore $G + H$ is Eulerian.

5. Give $C(K_{2,3})$, the completion of the complete bipartite graph with parts of order 2 and 3.

   Answer. We begin with $K_{2,3}$:



   We then add an edge between two vertices u and v if u and v are not adjacent, and their degree sum satisfies $\deg(u) + \deg(v) \geq 5$. So first we add an edge between r and s (each of which has degree 3), but not among x, y, and z (the sum of degrees will be 4). At this point, we are done. So $C(K_{2,3})$ is:



REMARK: Note that this graph is not Hamiltonian: arguing as in Problem 4, we see that a Hamiltonian cycle would necessarily include the edges xr, yr, and zr, which makes r incident on three edges from the putative cycle.

6. Let G be a connected r-regular graph of even order n such that $\overline{G}$ is also connected.

   (i) Prove that either G or $\overline{G}$ are Eulerian, but not both.

   Answer. Note that we must have $n \geq 4$, since the regular graphs of order 2 are either disconnected or have a disconnected complement.

   If r is even, then every vertex in G has even order, so G is Eulerian. But then $\overline{G}$ is $(n-1-r)$-regular; and since $n-1$ is odd and r is even, every vertex in $\overline{G}$ has odd order and there are at least four vertices, so $\overline{G}$ is not Eulerian.

   On the other hand, if r is odd then every vertex of G has odd order so G is not Eulerian; but every vertex in $\overline{G}$ has order $(n-1)-r$, which is the difference of two odd numbers and hence even. Since every vertex of $\overline{G}$ has even order, we know that $\overline{G}$ is Eulerian.

   (ii) Prove that either G or $\overline{G}$ are Hamiltonian.

   Answer. If $r \geq \frac{n}{2}$, then every vertex v satisfies $\deg(v) \geq \frac{n}{2}$, which implies that G is Hamiltonian by a theorem proven in class.

   On the other hand, if $r < \frac{n}{2}$, then $n-1-r > \frac{n}{2}-1$. And because n is even, that means that $\frac{n}{2}$ is an integer, so this means that $n-1-r \geq \frac{n}{2}$. Since $\overline{G}$ is $(n-1-r)$-regular, then the same theorem from the previous paragraph yields that $\overline{G}$ is Hamiltonian.

   In summary, either G is Hamiltonian (because r is large enough), or $\overline{G}$ is Hamiltonian (because $n-1-r$ is large enough).

   Note that it could be that both are Hamiltonian.