

Math 483 - Spring 26

HOMEWORK 9

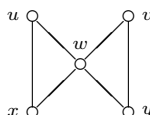
Solutions

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1. Give an example of a graph G such that:

- (i) G has an Eulerian circuit, but no Hamiltonian cycle (explain why it does not have a Hamiltonian cycle).

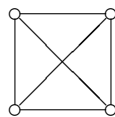
Answer. Consider the following graph of order 5:



Every vertex has even degree, so the graph has an Eulerian circuit. However, if we remove the vertex w , the graph is broken into two connected components. But if a graph is Hamiltonian, then removing k vertices results in a graph with at most k connected components, so the graph cannot have a Hamiltonian cycle.

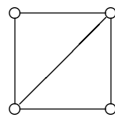
- (ii) G has a Hamiltonian cycle, but no Eulerian circuit (explain why it does not have an Eulerian circuit).

Answer. Note that K_4 is Hamiltonian, but every vertex is of degree 3; therefore, K_4 does not have an Eulerian circuit.



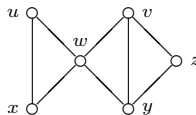
- (iii) G has a Hamiltonian cycle and an Eulerian trail, but no Eulerian circuit.

Answer. Add an edge to C_4 to obtain a graph with exactly two vertices of odd degree (so it has an Eulerian trail but no Eulerian circuit).



- (iv) G has neither an Eulerian circuit nor a Hamiltonian cycle, but does have an Eulerian trail.

Answer. The graph below does not have a Hamiltonian cycle because removing one vertex, namely w , results in a disconnected graph. It does not have an Eulerian circuit because it has two vertices of odd degree. But it does have an Eulerian trail, because it has exactly two vertices of odd degree.



An example of an Eulerian trail could be $(v, z, y, w, x, u, w, v, y)$.

2. Let G be a graph of order $n \geq 3$ such that for every pair of nonadjacent vertices $u \neq v$, we have $\deg(u) + \deg(v) \geq n - 1$. Prove that G contains a Hamiltonian path. HINT: Add a vertex x that is adjacent to every vertex of G and show that the resulting graph has a Hamiltonian cycle.

Proof. As the hint suggests, consider the graph H obtained by adding a new vertex x and making this vertex adjacent to every vertex in G . This is a graph of order $n + 1$; if u and v are nonadjacent vertices in H , $u \neq v$, then the two are vertices that are not adjacent in G and so $\deg_G(u) + \deg_G(v) \geq n - 1$. They are both also adjacent to x in H , so

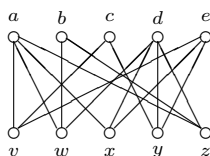
$$\deg_H(u) + \deg_H(v) = (\deg_G(u) + 1) + (\deg_G(v) + 1) \geq n - 1 + 2 = n + 1.$$

So in H , every pair of nonadjacent vertices has degree sum that is at least the order of the graph. By a theorem proven in class, this means that H has a Hamiltonian cycle.

We may consider the Hamiltonian cycle in H as starting and ending in x . Removing x will yield a path in G that includes every vertex of G . Thus, G has a Hamiltonian path.

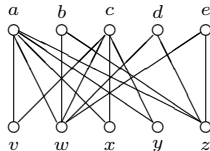
(It's possible that G also has a Hamiltonian cycle; the sufficient condition on degree sum is not necessary.)

3. Let G be the bipartite graph below with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. Determine if U can be matched to W :



Answer. Yes, we can match U to W : take the edges va , wb , xe , yc , and zd .

4. Let G be the bipartite graph below with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. Determine if U can be matched to W :



Answer. This graph does **not** have a matching of U and W . Consider the subset $X = \{v, x, y\}$ of U . Then $N(X) = \{a, c\}$. Since $|N(X)| < |X|$, Hall's condition tells us that there is no matching of U to W .

5. Show that a tree has at most one perfect matching.

Proof. Let T be a tree of order n . We will prove the result by induction on n .

Assume the result holds for trees of order less than n . Let v be an end-vertex of T , and let w be the only vertex that is adjacent to v . Any perfect matching of T will necessarily include the edge vw , because this is the only edge that covers v . Let $T' = T - \{v, w\}$. Because any perfect matching of T must include vw , perfect matchings of T' correspond to perfect matchings of T : from a perfect matching of T we get one of T' by removing vw ; and from a perfect matching of T' we obtain one for T by adding vw .

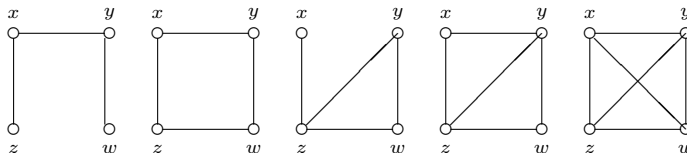
It follows that the number of perfect matchings in T is equal to the number of perfect matchings in T' , which by the inductive hypothesis is at most one. So T has at most one perfect matching, as claimed.

The argument above does not work for trees of order 1 or 2; for order 1, there is no perfect matching. For order 2, the unique edge is a perfect matching. In either case, we have at most one perfect matching, as required.

6. Show that every connected graph of order 4 has a perfect matching, except for $K_{1,3}$.

Proof. Clearly, $K_{1,3}$ has no perfect matching.

The remaining connected graphs of order 4 are:



They all have perfect matchings consisting of the edges xz and yw .

7. Let G be a connected graph of even order. Prove that if G contains no induced subgraph that is isomorphic to $K_{1,3}$, then G has a perfect matching.

Proof. This problem turned out to be harder than I anticipated. I apologize for that.

Assume we already know the result for graphs of even order strictly smaller than that of G with no induced subgraph isomorphic to $K_{1,3}$.

If G has a cut-vertex x , then the number of components of $G - x$ of odd order must be odd, and x is adjacent to each of the components. If there were more than one component of odd order, then there are at least 3 such components, G_i , $1 \leq i \leq 3$. Pick $x_i \in G_i$ that is adjacent to x ; then the subgraph induced by $\{x, x_1, x_2, x_3\}$ is isomorphic to $K_{1,3}$, which is impossible. Therefore, there is exactly one component of odd order, G_0 . The remaining components have even order strictly smaller than G and do not have an induced subgraph isomorphic to $K_{1,3}$, so they each have a perfect matching. And the subgraph induced by $V(G_0) \cup \{x\}$ is connected, has even order, no induced subgraph isomorphic to $K_{1,3}$, and order strictly smaller than that of G (since $G - x$ has at least two components). So G_0 also has a perfect matching. Putting all these perfect matchings together yields a perfect matching for G .

So assume that G does not have any cut vertex.

Let xy be an edge in G . If $G - \{x, y\}$ is connected, then it is a graph of even order strictly smaller than that of G , with no induced subgraphs isomorphic to $K_{1,3}$, and by the induction hypothesis it has a perfect matching M . Then $M \cup \{xy\}$ is a perfect matching for G .

If $G - \{x, y\}$ is disconnected, then the number of connected components of odd order is even. If the number is zero, then each connected component of $G - \{x, y\}$ has a perfect matching, and taking all these perfect matchings together with the edge $\{xy\}$ gives a perfect matching for G .

So we may assume instead that $G - \{x, y\}$ has at least two connected components of odd order.

If x is not adjacent to any vertex in one of the connected components, then there can be no path in $G - y$ from vertices in other components to vertices in the component that x is not adjacent to, which would mean that $G - x$ is disconnected; but we are assuming that there are no cut-vertices in G , so this is impossible. Therefore, x is adjacent to at least one vertex in each connected component of $G - \{x, y\}$. Symmetrically, so is y .

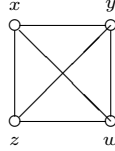
If either vertex from among x and y is adjacent to vertices in at least three connected components of odd order, then we get an induced subgraph isomorphic to $K_{1,3}$. So there must be exactly two connected components of odd order, and each of them has vertices adjacent to x and vertices adjacent to y . Call the two components G_1 and G_2 .

The subgraphs induced by $V(G_1) \cup \{x\}$ and $V(G_2) \cup \{y\}$ both are connected, and have even order strictly smaller than that of G . The components of even order of $G - \{x, y\}$ also have order strictly smaller than that of G . So by the induction hypothesis, each of these subgraphs have a perfect matching, which when put together give us a perfect matching for G .

Thus, in all situations, G has a perfect matching, as claimed.

8. Show that K_4 has two perfect matchings that are disjoint (no edge is contained in both matchings).

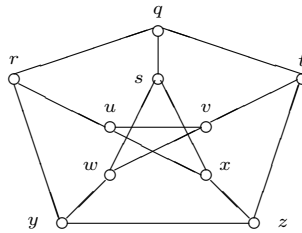
Answer. Take K_4 labeled as follows:



One perfect matching is given by $\{xy, zw\}$; a perfect matching that is disjoint from it is $\{xz, yw\}$.

9. Show that the Petersen graph has a perfect matching.

Answer. For reference, here is the Petersen graph:



A perfect matching in the Petersen graph is given by taking the five edges that connect the outer cycle with the inner cycle: $\{qs, tv, xz, wy, ru\}$.

This is not the only perfect matching for the Petersen graph: for example, we can also take $\{qs, ry, tz, ux, vw\}$.

10. Show that the Petersen graph does not contain two perfect matchings that are disjoint. HINT: Recall that the smallest cycle in the Petersen graph has length 5.

Answer. Let P denote the Petersen graph. Suppose we had two perfect matchings M and M' that are disjoint.

Consider the graph we get by taking the Petersen graph and removing the edges from M , $P - M$. Because the edges of M are independent, we are removing exactly one edge incident in each vertex. Since the Petersen graph is 3-regular, it follows that $P - M$ is 2-regular. In particular, the connected components of $P - M$ must be cycles, since a connected 2-regular graph is a cycle.

So we must have a collection of cycles whose lengths add up to 10, and they are all subgraphs of P . But the smallest cycle in P has length 5, so we must have that $M - P$ is the union of two cycles of length 5. And the matching M' must be a matching for $M - P$.

However, a cycle of length 5 does not have a perfect matching. So it is not possible for M' to be a matching contains in $P - M$.

Thus, P cannot have two perfect matchings that are disjoint.