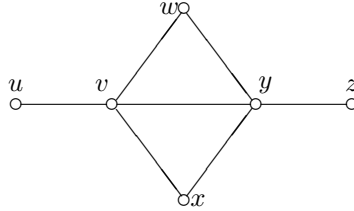


MATH 483 – Spring 2026  
**MIDTERM**  
 SOLUTIONS  
*Prof. Arturo Magidin*

1. Let  $G$  be the following graph:



(i) Give a  $u$ - $z$  walk that is not a trail. (2 points)

**Answer.** A walk that is not a trail is a walk that repeats edges. There are multiple possibilities here, and one of them would be  $u, v, y, x, v, y, z$ , which repeats the  $vy$  edge.

(ii) Give a  $u$ - $z$  trail that is not a path. (2 points)

**Answer.** A trail that is not a path is a walk that repeats vertices but does not repeat edges. Again there are multiple possible answers, one of them being  $u, v, w, y, v, x, y, z$ .

(iii) Give a  $u$ - $z$  path that is not a geodesic. (2 points)

**Answer.** This would be a walk that does not repeat vertices, but is not the shortest possible between  $u$  and  $z$ ; there are two possible answers:  $u, v, w, y, z$ , and  $u, v, x, y, z$ .

(iv) Give a  $u$ - $z$  geodesic. (2 points)

**Answer.** This is a walk of shortest possible length between  $u$  and  $z$ , which in this case is  $u, v, y, z$ .

2. Let  $G$  be a connected graph. Prove that an edge  $e \in E(G)$  is a bridge if and only if there is no cycle of  $G$  that includes  $e$ . (10 points)

**Proof.** Let  $e = uv$ . We prove both implications by contrapositive.

Assume first that  $e$  is in a cycle  $C$ ; that is, there is a cycle  $C = (u, v, u_2, \dots, u_n = u)$ . Now let  $x, y$  be vertices of  $G$ ,  $x \neq y$ . We want to show that there is an  $x$ - $y$  path that does not include  $e$  (so that  $G - e$  remains connected).

We know that there is an  $x$ - $y$  path  $P$  in  $G$ , because  $G$  is connected. If  $P$  does not include  $e$ , we are done. If  $P$  does include  $e$ , say

$$P = (x = w_0, w_1, \dots, w_i = v, u = w_{i+1}, \dots, w_k = y),$$

(if the path goes from  $u$  to  $v$ , then simply exchange the roles of  $x$  and  $y$ ). Then we can replace the edge from  $v$  to  $u$  with a walk around  $C$  to get a walk from  $x$  to  $y$ :

$$(x = w_0, \dots, w_i = v, u_2, u_3, \dots, u_n = u = w_{i+1}, \dots, w_k = y).$$

Since there is an  $x$ - $y$  walk in  $G - e$ , it follows that there is an  $x$ - $y$  path in  $G - e$ .

Thus, in either case there,  $x$  and  $y$  are connected in  $G - e$ . Since  $x$  and  $y$  were arbitrary, this means that  $G - e$  is connected, so  $e$  is not a bridge in  $G$ .

By contrapositive, this means that if  $e$  is a bridge in  $G$ , then it does not lie in any cycle.

Conversely, assume that  $e$  is not bridge. Then  $G - e$  is connected, so there is a  $v$ - $u$  path in  $G - e$ ,

$$P = (v = v_0, \dots, v_k = u).$$

Then  $(v = v_0, \dots, v_k = u, v)$  is a cycle in  $G$  that includes  $e$ . So if  $e$  is not a bridge, then there is a cycle that includes  $e$ . By contrapositive, if  $e$  is not in any cycle, then  $e$  is a bridge.

3. Determine if each of the following sequences is graphical. You may invoke the Havel-Hakimi theorem to reduce the problem to a different sequence, or present a different argument (or a combination). To complete the determination, either explain why the last sequence you are considering is not graphical, or draw a graph that has that degree sequence to show that it is graphical. (4 points each; 8 points total)

- (i)  $s_1$ : 5, 4, 4, 3, 2, 2.

**Answer.** Applying the Havel-Hakimi Theorem,  $s_1$  is graphical if and only if the sequence 3, 3, 2, 1, 1 is graphical (delete the 5, and subtract one from each of the following five numbers on the list).

Again applying the Havel-Hakimi Theorem, this sequence is graphical if and only if the sequence 2, 1, 0, 1 (or equivalently, the sequence 2, 1, 1, 0) is graphical.

This last sequence is graphical: for example, it is the degree sequence of the following graph:



So  $s_1$  is also graphical.

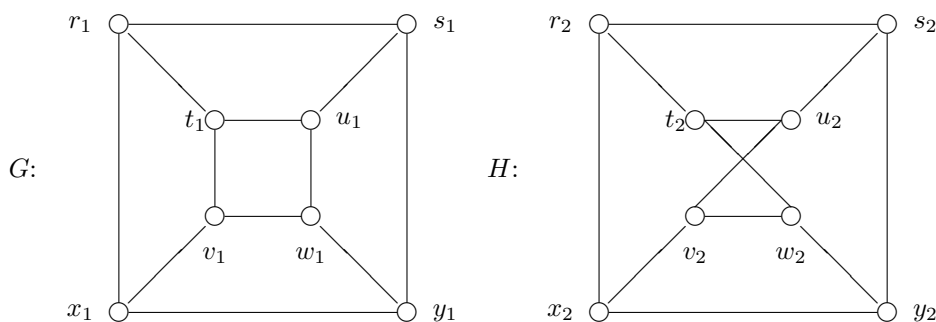
- (ii)  $s_2$ : 6, 5, 5, 4, 3, 2, 1.

**Answer.** Again applying the Havel-Hakimi Theorem, this sequence is graphical if and only if the sequence 4, 4, 3, 2, 1, 0 is graphical.

This sequence is graphical if and only if the sequence 3, 2, 1, 0, 0 is graphical. But this sequence cannot be graphical: it would require one vertex to be adjacent to three other vertices, but there are only a total of 3 vertices that have edges incident on them. This is impossible.

If you do not spot this problem, we can apply the Havel-Hakimi Theorem again to note that the sequence 3, 2, 1, 0, 0 is graphical if and only if 1, 0, -1, 0 is graphical. Since this sequence contains negative numbers, it is certainly not graphical, so  $s_2$  is also not graphical.

4. Consider the following two graphs:



Determine whether  $G$  is isomorphic to  $H$  or not. If you state they are isomorphic, explicitly describe the assignment  $V(G) \rightarrow V(H)$  that realizes the isomorphism. If you state they are not isomorphic, justify your answer. (8 points)

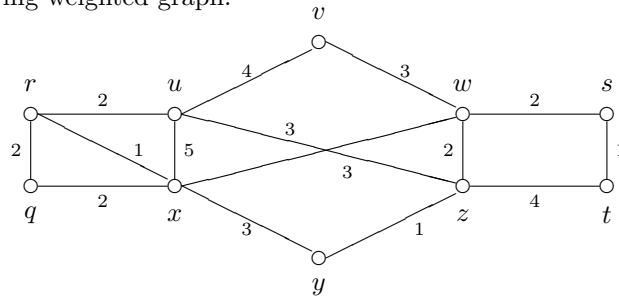
**Answer.** The graphs are **not** isomorphic.

Note that  $G$  is bipartite: if we divide the vertices into the sets  $U = \{r_1, u_1, v_1, y_1\}$  and  $V = \{s_1, t_1, w_1, x_1\}$ , then we see that all edges of  $G$  join a vertex in  $U$  with a vertex in  $V$ .

On the other hand,  $H$  contains odd cycles: for example,  $(r_2, t_2, w_2, v_2, x_2, r_2)$  is a cycle of length 5. Since a graph is bipartite if and only if it does not contain any odd cycles, we conclude that  $H$  is not bipartite.

So we can conclude that  $G$  and  $H$  cannot be isomorphic, because  $G$  is bipartite and  $H$  is not.

5. Let  $G$  be the following weighted graph:



In the space below, draw the edges of a minimum spanning tree for  $G$ . (4 points)

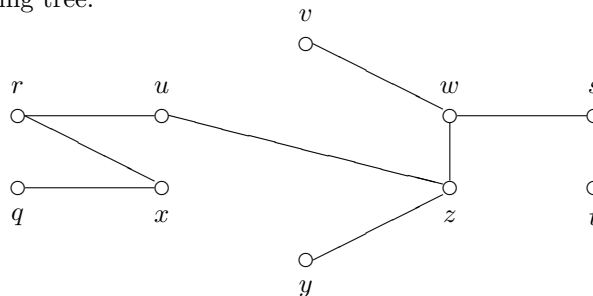
**Answer.** There are a number of possibilities here at some points.

Using Kruskal's algorithm, we would first add the edges of weight 1:  $rx$ ,  $yz$ , and  $st$ .

Then we want to add edges of weight 2 that do not create cycle: we will add  $wz$  and  $ws$ , and we will add  $ru$ . Then we do come to a decision point, because we must add **either**  $rq$  **or**  $qx$ , but not both. Let's say we add  $qx$ .

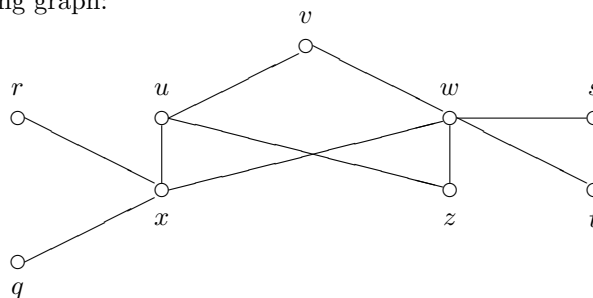
Then we continue with edges of weight 3. We add  $vw$ ; and we will also add one of  $xy$ ,  $xw$ , or  $uz$ , but not two or all three. We cannot add both  $xy$  and either  $xw$  or  $uz$ , because that will create a cycle; and similarly with  $uz$  and  $xw$ . Say we add  $uz$ .

At this point we have added nine edges to a graph with ten vertices, so we are done and this is a minimum spanning tree:



As mentioned above, other possibilities exist, depending on which of the two edges  $rq$  and  $rx$  you pick; and which of the edges  $xy$ ,  $xw$ , and  $uz$  you choose.

6. Let  $G$  be the following graph:



- (i) List all the bridges of  $G$ . (2 points)

**Answer.** The bridges are the edges  $rx$  (which would separate  $r$ ); and  $qx$  (which would separate  $q$ ). Those are the only ones, since all other edges lie in at least one cycle.

- (ii) List all the cut-vertices of  $G$ . (2 points)

**Answer.** The cut vertices are  $x$  and  $w$ .

- (iii) List all the blocks of  $G$  (4 points)

**Answer.** We have four blocks. They are:

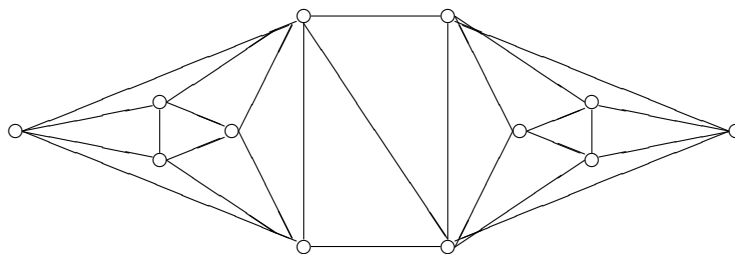
$$B_1 = \{r, x\},$$

$$B_2 = \{q, x\},$$

$$B_3 = \{u, v, w, x, z\}$$

$$B_4 = \{w, s, t\}.$$

7. For the graph  $G$  below:



Determine each of the following quantities; (1 points each, 4 points total)

- (i) The vertex connectivity  $\kappa(G)$ .

**Answer.** The graph can be disconnected by removing two vertices: for instance, any of the two vertices on the same side of the large central rectangle. But it cannot be disconnected by removing a single vertex, so  $\kappa(G) = 2$ .

- (ii) The edge connectivity  $\lambda(G)$ .

**Answer.** The graph can be disconnected by removing three edges; for example, the two horizontal edges and the diagonal edge in the central rectangle. But there are no two edges we could remove to disconnect the graph. So  $\lambda(G) = 3$ .

- (iii) The minimum degree  $\delta(G)$ .

**Answer.** The minimum degree is 4, which is achieved by any of the eight vertices that are not in the large central rectangle; so  $\delta(G) = 4$ .

- (iv) The maximum degree  $\Delta(G)$ .

**Answer.** The maximum degree is achieved by the top left and the bottom right vertices of the large central rectangle; it is 6. So  $\Delta(G) = 6$ .