Some basic cardinal arithmetic

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The precise definitions involved in cardinal arithmetic can be somewhat technical, and for our purposes not terribly important. To develop it completely one usually needs to start with ORDINALS and ORDINAL ARITHMETIC. There are also some set-theoretical issues that need to be taken care of.

If you want a very good introduction, I strongly recommend Paul Halmos's *Naive Set Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, 1974, ISBN 0-387-90092-6; cardinal arithmetic is in Chapter 25, but you would need to look at several previous chapters as well. There is also a basic exposition in Thomas W. Hungerford's *Algebra*, Graduate Texts in Mathematics vol. 73, Springer-Verlag, 1980, ISBN 0-387-90518-9, Section 0.8.

If you want a truly exhaustive (and exhausting) exposition, showing all the guts of the formal underlying set theory, you can look at Thomas Jech's *Set Theory*, 2nd Corrected Edition, Perspective in Mathematical Logic, Springer-Verlag, 1997, ISBN 3-540-63048-1, Section 1.6.

In Zermelo-Fraenkel Set Theory, the natural numbers are defined by letting 0 represent $\varnothing$; if a is a set, then the successor of a is $s(a) = a \cup \{a\}$, and so we define $1 = s(0)$, $2 = s(1)$, etc.

A set X is "inductive" if $\varnothing \in X$ and for all x, if $x \in X$ then $s(x) \in X$. The Axiom of Infinity guarantees the existence of at least one inductive set, and the set of all natural numbers $\mathbb{N}$ can then be shown to be the smallest inductive set (in the sense that it is inductive and contained in every inductive set). In this definition, note that each natural number n, *as a set*, is equal to the set of all smaller natural numbers, so that $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, and so on. Note that a nice feature of this definition is that the set n has exactly n elements. In addition, since $0 \in 1$, $1 \in 2$, $2 \in 3$, etc., the set of all natural numbers is ordered with respect to the relation $\in$.

Definition 1. A set X is said to be FINITE if there is a bijection $f\colon X \to n$ for $n \in \mathbb{N}$. A set that is not finite is INFINITE.

There are other ways to define finite and infinite; for example, one definition is that a set X is "Dedekind-finite" if there is no bijection between X and a proper subset of X, and is "Dedekind-infinite" if there is a bijection between X and a proper subset of X. If we assume the Axiom of Choice, then a set is Dedekind-finite if and only if it is finite under the above definition, and a set is Dedekind-infinite if and only if it is infinite in the sense of the definition above. However, without the Axiom of Choice, there may be sets that are Dedekind-finite but not finite.

Definition 2. Two sets A and B are **equipollent** ("equally powerful") if and only if there exists a bijection $f\colon A \to B$. In that case, we write $A \sim B$.

This is an equivalence relation, but because the class of all sets is not a set, we cannot use something like the Fundamental Theorem of Equivalence Relations (which connects equivalence relations on a set X with partitions of X). However, we will pretend this is not an issue and that we can take "equivalence classes" under equipollence. In practice, what one must do is to describe a special class of sets (called the *ordinal numbers*, which are a generalization of natural numbers and that are well-ordered amongst themselves), then prove that every set is equipollent to at least one ordinal, and then show the set of all ordinals that are equipollent to a set X has a "smallest" ordinal, which cannot be bijected with any smaller ordinal. These ordinals are then singled out as the "cardinal numbers". But we will use a non-formalized, naive version of this:

Definition 3. The CARDINAL NUMBER or CARDINALITY of a set A, denoted $|A|$, is the equivalence class of A under equipollence. We say $|A|$ is an infinite cardinal if and only if A is an infinite set, and $|A|$ is a finite cardinal if and only if A is a finite set.

Cardinal numbers are commonly denoted with lowercase Greek letters, usually starting around κ, so κ, λ, μ, etc.

Proposition 4. *The following hold:*

(i) *Every set has a unique cardinal number.*
(ii) *Two sets have the same cardinal number if and only if they are equipollent.*
(iii) *The cardinal number of a finite set is the number of elements in the set.*

The cardinal number of the set $\mathbb{N}$ of natural numbers is called "aleph-zero" or "aleph-naught", denoted $\aleph_0$ (aleph is the first letter of the Hebrew alphabet). It can be shown that given any set of cardinals, there is a smallest cardinal strictly larger than the ones in the set. The first cardinal after $\aleph_0$ is $\aleph_1$; then you have $\aleph_2$, $\aleph_3, \ldots, \aleph_n, \ldots$. The smallest cardinal strictly larger than this is $\aleph_\omega$ ("aleph-omega"), and more. Alephs are indexed by ordinal numbers.

Definition 5. Let κ and μ be cardinal numbers. The **sum** $\kappa+\mu$ is defined to be $|(A\times\{0\})\cup(B\times\{1\})|$, where $|A| = \kappa$ and $|B| = \mu$.

That is: $\kappa+\mu$ is the cardinality of the disjoint union of a set of cardinality κ and a set of cardinality μ.

Definition 6. Let κ and μ be cardinal numbers. The **product** $\kappa\mu$ is defined to be $|A \times B|$, where $|A| = \kappa$ and $|B| = \mu$.

Definition 7. Let κ and μ be cardinal numbers, and let A and B be sets, with $|A| = \kappa$, $|B| = \mu$. We say $\kappa \leq \mu$ if there is an injection $f\colon A \to B$.

Naturally, $\kappa < \mu$ means ($\kappa \leq \mu$ and $\kappa \neq \mu$).

Theorem 8 (Cantor's Theorem)**.** *Let A be a set. Then $|A| < |P(A)|$.*

Proof. Let $f\colon A \to P(A)$ be any function. We show that f is not surjective. Let $B = \{x \in A \mid x \notin f(x)\}$. Then $B \in P(A)$, but for every $a \in A$, if $a \in B$ then $a \notin f(a)$, hence $f(a) \neq B$; and if $a \notin B$, then $a \in f(a)$ hence $f(a) \neq B$. Either way, $f(a) \neq B$. Thus, $B \notin \text{Image}(f)$, so f is not onto.

Thus, $|A| \neq |P(A)|$. On the other hand, $f\colon A \to P(A)$ given by $f(a) = \{a\}$ is one-to-one, so $|A| \leq |P(A)|$. This proves the desired inquality. □

Theorem 9 (Cantor-Schroeder-Bernstein Theorem)**.** *If A and B are sets, $|A| \leq |B|$, and $|B| \leq |A|$, then $|A| = |B|$.*

Theorem 10 (Trichotomy)**.** *For all cardinals κ, λ, exactly one of the following is true:*

$$\kappa < \lambda, \qquad \kappa = \lambda, \qquad \lambda < \kappa.$$

Theorem 11. *The collection of cardinals is well-ordered: every nonempty set of cardinals have a smallest element.*

A surprising fact of cardinal arithmetic is the following:

Theorem 12. *Let κ and μ be cardinals.*

(i) *If at least one of κ and μ is infinite, then $\kappa + \mu = \max\{\kappa, \mu\}$.*
(ii) *If neither κ nor μ are zero, and at least one is infinite, then $\kappa\mu = \max\{\kappa, \mu\}$.*
(iii) *In particular, if at least one of κ and μ is infinite, and neither is zero, then $\kappa + \mu = \kappa\mu = \max\{\kappa, \mu\}$.*

Some other facts:

Definition 13. Let κ and λ be cardinals. Then κ^λ is the cardinality $|A^B|$, where $|A| = \kappa$, $|B| = \lambda$, and $Y^X = \{f\colon X \to Y \mid f \text{ is a function}\}$.

Theorem 14. (i) *If A is a set and n is a positive integer, then* $|A^n| = |\underbrace{A \times \cdots \times A}_{n \text{ factors}}|$.

(ii) *If A is finite and n is a positive integer, then $|A^n| = |A|^n$. If A is infinite, then $|A^n| = |A|$.*

(iii) $\left| \bigcup_{n \geq 0} A^n \right| = \aleph_0 |A|$.

Corollary 15. *If A is an infinite set, and $F(A)$ is the set of all finite subsets of A, then $|F(A)| = |A|$.*

Corollary 16. *If α is a cardinal, and for every $i \in I$, $|A_i| \leq \alpha$, then $|\cup_{i \in I} A_i| \leq |I|\alpha$.*

Cardinal arithmetic can be somewhat unintuitive; in particular, it is generally very hard to show strict inequalities when comparing sums and differences, even if given strict inequalities between the summands. For example, one can have two families of cardinals, κ_i and μ_i with $i \in I$, that satisfy $\kappa_i < \mu_i$ for all i, and yet have $\sum \kappa_i = \sum \mu_i$.

One notable exception to this is König's Theorem:

Theorem 17 (König's Theorem)**.** *Let I be a set, and let $\{\kappa_i\}_{i \in I}$ and $\{\mu_i\}_{i \in I}$ be two families of cardinals, with $\kappa_i < \mu_i$ for each $i \in I$. Then*

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \mu_i.$$

It is not hard to show that $|\mathbb{R}| = |P(\mathbb{N})|$. That is, there is a bijection between the set of real numbers and the power set of the natural numbers. From Cantor's Theorem, we know that $|P(\mathbb{N})| > \aleph_0$. Cantor believed that $|\mathbb{R}|$ would be the "next" cardinal, but was unable to prove it. This is known as the Continuum Hypothesis. Now, $|P(\mathbb{N})| = 2^{\aleph_0}$, by identifying a subset with its characteristic function. Written this way, we can phrase the Continuum Hypothesis as:

Continuum Hypothesis (CH). $2^{\aleph_0} = \aleph_1$.

The first problem in Hilbert's famous list from his 1900 address at the International Congress of Mathematicians was to determine the truth or falsity of the Continuum Hypothesis.

In 1940, Kurt Gödel showed that it was impossible to disprove the Continuum Hypothesis in ZF Set Theory. In 1964, Paul Cohen proved that it was impossible to *prove* the Continuum Hypothesis in ZF Set Theory, even including the Axiom of Choice. that is, the Continuum Hypothesis is **independent** of the other Axioms of ZF Set Theory and of ZFC Set Theory.

A generalization of the Continuum Hypothesis is:

General Continuum Hypothesis (GCH). $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α.

Like the Continuum Hypothesis, the Generalized Continuum Hypothesis is independent of the axioms of ZF Set Theory, or even ZFC Set Theory. However, the Generalized Continuum Hypothesis is known to *imply* the Axiom of Choice in ZF Set Theory.