Math 561 - Homework 1
Due Wednesday September 9

1. Let $\mathbf{V}$ be a vector space. We saw in class that the intersection of any family of subspaces of $\mathbf{V}$ is again a subspace of $\mathbf{V}$. Give an example of a vector space $\mathbf{V}$, and two subspaces $\mathbf{W}_1$ and $\mathbf{W}_2$ of $\mathbf{V}$, such that $\mathbf{W}_1 \cup \mathbf{W}_2$ is **not** a subspace of $\mathbf{V}$. You do not need to prove that $\mathbf{W}_1$ and $\mathbf{W}_2$ are subspaces (but be sure they are), but you must explicitly show that $\mathbf{W}_1 \cup \mathbf{W}_2$ is not a subspace.

2. In fact, prove that if $\mathbf{W}_1$ and $\mathbf{W}_2$ are subspaces of a vector space $\mathbf{V}$, then $\mathbf{W}_1 \cup \mathbf{W}_2$ is also a subspace if and only if either $\mathbf{W}_1 \subseteq \mathbf{W}_2$, or $\mathbf{W}_2 \subseteq \mathbf{W}_1$.

Definition. If S_1 and S_2 are two nonempty subsets of a vector space $\mathbf{V}$, then the *sum* of S_1 and S_2, denoted $S_1 + S_2$, is the set $S_1 + S_2 = \{x + y \mid x \in S_1 \text{ and } y \in S_2\}$.

3. Show that if $\mathbf{W}_1$ and $\mathbf{W}_2$ are subspaces, then $\mathbf{W}_1 + \mathbf{W}_2$ is a subspace of $\mathbf{V}$, and in fact is the smallest subspace of $\mathbf{V}$ that contains $\mathbf{W}_1$ and $\mathbf{W}_2$.

4. Show that if S_1 and S_2 are arbitrary subsets of a vector space $\mathbf{V}$, then
$$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2).$$
Give an example in which $\operatorname{span}(S_1 \cap S_2)$ and $\operatorname{span}(S_1) \cap \operatorname{span}(S_2)$ are equal, and one in which they are unequal.

5. Show that if S_1 and S_2 are subsets of a vector space $\mathbf{V}$, then
$$\operatorname{span}(S_1 \cup S_2) = \operatorname{span}(S_1) + \operatorname{span}(S_2).$$

6. Show that for any subset S of $\mathbf{V}$, $\operatorname{span}(\operatorname{span}(S)) = \operatorname{span}(S)$.

7. Let S be a finite subset of $\mathbf{V}$ with exactly k elements. Prove that S is linearly independent if and only if $\dim(\operatorname{span}(S)) = k$.

8. Let $\mathbf{V}$ be a vector space. Prove that a subset S of $\mathbf{V}$ is linearly independent if and only if there exists a subspace $\mathbf{W}$ of $\mathbf{V}$ such that S is a basis for $\mathbf{W}$.

9. We proved the following statements are true in the case of finite dimensional vector spaces. Use $\mathbf{V} = \mathbb{R}[x]$, the vector space of all polynomials with real coefficients (vector space over $\mathbb{R}$, of dimension $\aleph_0$) to give explicit counterexamples to each of them. Here, $|X|$ means the cardinality of the set X.

   (i) If $S \subseteq \mathbf{V}$ is a generating set with $|S| = \dim(\mathbf{V})$, then S is a basis for $\mathbf{V}$.
   (ii) If $L \subseteq \mathbf{V}$ is a linearly independent set with $|L| = \dim(\mathbf{V})$, then L is a basis for $\mathbf{V}$.
   (iii) If $\mathbf{W}$ is a subspace of $\mathbf{V}$, and $\dim(\mathbf{W}) = \dim(\mathbf{V})$, then $\mathbf{W} = \mathbf{V}$.