

Math 561 - Homework 2
Due Wednesday September 16

1. Let $\mathbf{V}$ be a vector space, $\mathbf{V} \neq \{\mathbf{0}\}$. Prove that $\dim(\mathbf{V}) = 1$ if and only if every nonzero vector in $\mathbf{V}$ is a basis for $\mathbf{V}$.
2. Recall that if $\mathbf{V} = \{a \in \mathbb{R} \mid a > 0\}$, then $\mathbf{V}$ is a real vector space using the operations $\oplus$ and $\odot$, defined as:

$$r \oplus s = rs \text{ for all } r, s \in \mathbf{V},$$
$$\alpha \odot r = r^\alpha \text{ for all } r \in \mathbf{V}, \alpha \in \mathbb{R}.$$

Prove that if $r \in \mathbf{V}$, then $\{r\}$ is a basis for $\mathbf{V}$ if and only if $r \neq 1$. Conclude that $\dim(\mathbf{V}) = 1$.

3. Let X be a set. Recall that $\mathbb{R}^X$, also denoted $\mathcal{F}(X, \mathbb{R})$, is the vector space of all functions $f: X \rightarrow \mathbb{R}$ under pointwise addition and scalar multiplication. That is, if $f, g \in \mathcal{F}(X, \mathbb{R})$ and $a \in \mathbb{R}$, then we define $f + g$ to be the function given by $(f + g)(x) = f(x) + g(x)$, where the addition on the right hand side is the usual addition of real numbers; and $a \cdot f$ is the function defined by $(a \cdot f)(x) = a f(x)$, where the operation on the right hand side is the usual multiplication of real numbers.

Define $\mathbb{R}^{(X)}$ be the subset of $\mathbb{R}^X$ given by

$$\mathbb{R}^{(X)} = \{f: X \rightarrow \mathbb{R} \mid f(x) = 0 \text{ for almost all } x \in X\},$$

where “almost all” means “all except perhaps for finitely many.”

- (i) Prove that $\mathbb{R}^{(X)}$ is a subspace of $\mathcal{F}(X, \mathbb{R})$.
 - (ii) Prove that $\dim(\mathbb{R}^{(X)}) = |X|$, the cardinality of the set X . HINT: Construct a basis that is indexed by elements of X .
4. Let $\mathbf{V}$ be a vector space, and let $\mathbf{W}_1$ and $\mathbf{W}_2$ be finite dimensional subspaces of $\mathbf{V}$. Prove that

$$\dim(\mathbf{W}_1 \cap \mathbf{W}_2) + \dim(\mathbf{W}_1 + \mathbf{W}_2) = \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2).$$

HINT: Start with a basis for $\mathbf{W}_1 \cap \mathbf{W}_2$, and extend it to a basis for $\mathbf{W}_1$ and to a basis for $\mathbf{W}_2$.

5. Let $\mathbf{V}$ be a finite dimensional vector space, and let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\mathbf{V}$. Let α be a scalar, and i, j be integers, $1 \leq i, j \leq n$, $i \neq j$. Prove that if you replace $\mathbf{v}_i$ with $\mathbf{v}_i + \alpha \mathbf{v}_j$ in β , the resulting set is still a basis for $\mathbf{V}$.
6. Does the equality in Problem 4 hold if we do not assume that $\mathbf{W}_1$ and $\mathbf{W}_2$ are finite dimensional? Prove or give a counterexample. SUGGESTION: Recall that if κ and λ are cardinals, and at least one of them is infinite, then $\kappa + \lambda = \max(\kappa, \lambda)$.