

Math 561 - Homework 3
Due Wednesday September 23

NOTE: There are six problems in this homework; three problems are on the back of the page.

1. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over $\mathbb{F}$. Let $\mathbf{V} \times \mathbf{W}$ be the set

$$\mathbf{V} \times \mathbf{W} = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in \mathbf{V}, \mathbf{w} \in \mathbf{W}\},$$

and define addition and scalar multiplication coordinate-wise; that is,

$$\begin{aligned}(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) &= (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2), \\ \alpha(\mathbf{v}, \mathbf{w}) &= (\alpha\mathbf{v}, \alpha\mathbf{w}).\end{aligned}$$

This makes $\mathbf{V} \times \mathbf{W}$ into a vector space. (You may take this for granted, but if you have not seen this proven before I encourage you to prove it for yourself). This vector space is called the (DIRECT) PRODUCT OF $\mathbf{V}$ AND $\mathbf{W}$.

- (i) Prove that if $\beta = \{\mathbf{v}_i\}_{i \in I}$ and $\gamma = \{\mathbf{w}_j\}_{j \in J}$ are bases for $\mathbf{V}$ and $\mathbf{W}$, respectively, then

$$\{(\mathbf{v}_i, \mathbf{0}_\mathbf{W})\}_{i \in I} \cup \{(\mathbf{0}_\mathbf{V}, \mathbf{w}_j)\}_{j \in J}$$

is a basis for $\mathbf{V} \times \mathbf{W}$.

- (ii) Express $\dim(\mathbf{V} \times \mathbf{W})$ in terms of $\dim(\mathbf{V})$ and $\dim(\mathbf{W})$.

2. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over $\mathbb{F}$, and let $\mathbf{V} \times \mathbf{W}$ be the direct product of $\mathbf{V}$ and $\mathbf{W}$. Let $\pi_1: \mathbf{V} \times \mathbf{W} \rightarrow \mathbf{V}$ be the function $\pi_1(\mathbf{v}, \mathbf{w}) = \mathbf{v}$, and $\pi_2: \mathbf{V} \times \mathbf{W} \rightarrow \mathbf{W}$ be the function $\pi_2(\mathbf{v}, \mathbf{w}) = \mathbf{w}$.

- (i) Show that π_1 and π_2 are linear transformations.

- (ii) Show that given any vector space $\mathbf{Z}$ over $\mathbb{F}$ and linear transformations $T_1: \mathbf{Z} \rightarrow \mathbf{V}$ and $T_2: \mathbf{Z} \rightarrow \mathbf{W}$, there exists a unique linear transformation $U: \mathbf{Z} \rightarrow \mathbf{V} \times \mathbf{W}$ such that $\pi_1 \circ U = T_1$ and $\pi_2 \circ U = T_2$.

3. Let $\mathbf{V} = \mathbb{R}^\infty$ be the vector space of real sequences with pointwise addition and scalar multiplication. Define $\rho, \lambda: \mathbf{V} \rightarrow \mathbf{V}$ to be the “right shift” and “left shift” functions:

$$\begin{aligned}\rho(a_1, a_2, \dots) &= (0, a_1, a_2, \dots) \\ \lambda(a_1, a_2, \dots) &= (a_2, a_3, \dots).\end{aligned}$$

- (i) Show that ρ and λ are linear transformations.

- (ii) Show that λ is onto, but not one-to-one.

- (iii) Show that ρ is one-to-one, but not onto.

- (iv) Show that $\lambda \circ \rho$ is the identity linear transformation ($(\lambda \circ \rho)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbf{V}$); but that $\rho \circ \lambda$ is not the identity linear transformation.

4. Let $T: \mathbf{P}_3(\mathbb{R}) \rightarrow \mathbf{M}_{2 \times 2}(\mathbb{R})$ be given by

$$T(p(x)) = \begin{pmatrix} p(1) & p'(0) \\ p(2) & p(0) \end{pmatrix}.$$

- (i) Prove that T is a linear transformation.
- (ii) Find $\mathbf{N}(T)$ and $\mathbf{R}(T)$.

5. Let $T: \mathbf{V} \rightarrow \mathbf{W}$ be a linear transformation. Prove that the following two statements are equivalent:

- (i) T is one-to-one.
- (ii) If $S \subseteq \mathbf{V}$ is linearly independent in $\mathbf{V}$, then $T(S) = \{T(s) \mid s \in S\}$ is linearly independent in $\mathbf{W}$.

That is, T is one to one if and only if it sends linearly independent sets to linearly independent sets.

6. Let $T: \mathbf{V} \rightarrow \mathbf{W}$ be a linear transformation. Prove that the following two statements are equivalent:

- (i) T is surjective.
- (ii) If $S \subseteq \mathbf{V}$ generates $\mathbf{V}$, then $T(S)$ generates $\mathbf{W}$.

That is, T is surjective if and only if it sends generating sets to generating sets.