

Math 561 - Homework 4
Due Wednesday September 30

1. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces, not necessarily finite dimensional. Prove that a linear transformation $T: \mathbf{V} \rightarrow \mathbf{W}$ is invertible if and only if for every subset S of $\mathbf{V}$, if S is a basis for $\mathbf{V}$, then $T(S)$ is a basis for $\mathbf{W}$.
2. Let β be the following basis for $\mathbb{R}^4$:

$$\beta = \left\{ (1, 0, 0, 0), \quad (1, 1, 0, 0), \quad (1, 1, 1, 0), \quad (1, 1, 1, 1) \right\}.$$

You may take for granted that β is a basis. Let T be the (unique) linear transformation $T: \mathbb{R}^4 \rightarrow \mathbf{P}_2(\mathbb{R})$, such that $T(1, 0, 0, 0) = 1 - x$, $T(1, 1, 0, 0) = 2x - x^2$, $T(1, 1, 1, 0) = \frac{1}{2}x + \frac{1}{3}x^2$, and $T(1, 1, 1, 1) = 0$.

Give a formula for $T(a, b, c, d)$ in terms of a, b, c , and d .

3. Let $T: \mathbf{P}_3(\mathbb{R}) \rightarrow \mathbf{M}_{2 \times 2}(\mathbb{R})$ be given by

$$T(p(x)) = \begin{pmatrix} p(1) & p'(0) \\ p(2) & p(0) \end{pmatrix}.$$

We proved in Homework 3 that T is a linear transformation. Let $\beta = [1, x, x^2, x^3]$ be the standard ordered basis for $\mathbf{P}_3(\mathbb{R})$, and let $\gamma = [E_{11}, E_{12}, E_{21}, E_{22}]$ be an ordered basis for $\mathbf{M}_{2 \times 2}(\mathbb{R})$.

- (i) Find $[T]_{\beta}^{\gamma}$.
 - (ii) Use (i) to find $T(2 - \frac{3}{2}x + 5x^2 - x^3)$.
4. Let $a \in \mathbb{R}$ be a real number, and define $T_a: \mathbf{P}_2(\mathbb{R}) \rightarrow \mathbf{P}_2(\mathbb{R})$ to be given by $T_a(p(x)) = p(x - a)$. Let β be the standard ordered basis for $\mathbf{P}_2(\mathbb{R})$.
 - (i) Find $[T_a]_{\beta}^{\beta}$.
 - (ii) Find the value of c for which $T_a \circ T_b = T_c$.
 - (iii) Use (ii) to find $[T_a]_{\beta}^{\beta}[T_b]_{\beta}^{\beta}$ without computing the matrix product explicitly.
 - (iv) Find $([T_a]_{\beta}^{\beta})^{-1}$ without computing the matrix inverse.
 5. Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces, and suppose that $T: \mathbf{V} \rightarrow \mathbf{W}$ and $U: \mathbf{W} \rightarrow \mathbf{V}$ are linear transformations.
 - (i) Prove that if $\dim(\mathbf{V}) = \dim(\mathbf{W})$ and $UT = I_{\mathbf{V}}$, the identity on $\mathbf{V}$, then $U = T^{-1}$.
 - (ii) Give an example to show that (i) need not hold if we do not assume $\dim(\mathbf{V}) = \dim(\mathbf{W})$.
 - (iii) Give an example to show that (i) need not hold if $\dim(\mathbf{V}) = \dim(\mathbf{W})$ but we do not assume that the dimensions are finite.
 6. Show that if A and B are $n \times n$ matrices and $AB = I_n$, then $A = B^{-1}$.
 7. Give an example of matrices A and B such that $AB = I_2$ the 2×2 identity matrix, but $A \neq B^{-1}$.
 8. Let $\beta = [(1, 1, 1), (1, 0, 1), (1, 1, 2)]$; you may take for granted that β is a basis for $\mathbb{R}^3$. Let A be the matrix

$$A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}.$$

Find $[L_A]_{\beta}^{\beta}$.