

Math 561 - Homework 5

Due Wednesday October 21

1. Show that a function $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ is a linear transformation if and only if there exist functionals $f_1, \dots, f_m \in (\mathbb{F}^n)^*$ such that

$$T(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$$

for all $\mathbf{x} \in \mathbb{F}^n$.

2. Let $\mathbf{V} = \mathbf{P}_n(\mathbb{F})$ and let $c_0, c_1, \dots, c_n$ be distinct scalars in $\mathbb{F}$.

- (i) For $0 \leq i \leq n$, define $f_i \in \mathbf{V}^*$ by $f_i(p) = p(c_i)$. Prove that $[f_0, \dots, f_n]$ is a basis for $\mathbf{V}^*$.
HINT: Apply any linear combination of these functionals that is equal to the zero function to the polynomial $p(x) = (x - c_1) \cdots (x - c_n)$ to deduce the coefficient of f_0 in the linear combination is zero.
- (ii) Conclude that there are unique polynomials $p_0, \dots, p_n \in \mathbf{V}$ such that $p_i(c_j) = \delta_{ij}$ (Kronecker's delta).
- (iii) For any scalars $a_0, \dots, a_n$, not necessarily distinct, deduce the existence of a unique polynomial $q(x)$ of degree at most n such that $q(c_i) = a_i$ for $i = 0, 1, \dots, n$. In fact,

$$q(x) = \sum_{i=0}^n a_i p_i(x).$$

- (iv) Deduce the Lagrange Interpolation Formula: for all $p(x) \in \mathbf{V}$,

$$p(x) = \sum_{i=0}^n p(c_i) p_i(x).$$

- (v) For each i , $i = 0, 1, \dots, n$, define

$$f_i(x) = \frac{(x - c_0) \cdots (x - c_{i-1})(x - c_{i+1}) \cdots (x - c_n)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)} = \prod_{\substack{k=0 \\ k \neq i}}^n \frac{x - c_k}{c_i - c_k}.$$

Verify that $f_i(c_j) = \delta_{ij}$, and conclude that $f_i = p_i$, where p_i are the polynomials whose existence was established in (ii).

- (vi) Find the unique polynomial of degree at most 4 such that $p(0) = 1$, $p(1) = 2$, $p(2) = 4$, $p(3) = 8$, and $p(4) = 16$.
- (vii) Find $p(5)$, the value of the polynomial $p(x)$ you found in (vi) at $x = 5$.
3. Find the dimension of, and a basis for, the solution space of

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 0 \\ 3x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 0. \end{array}$$

4. Find all solutions to the system of linear equations

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 1 \\ 3x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 3. \end{array}$$

5. Determine whether the following system of linear equations has a solution. You do not need to find the solutions if they exist, just determine whether there is a solution or not.

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 3 \\ x_1 & - & 4x_2 & + & 7x_3 & = & 4. \end{array}$$