

Math 561 - Homework 6

Due Wednesday October 28

1. Let $\mathbf{V}$ be a vector space and let $T: \mathbf{V} \rightarrow \mathbf{V}$ be a linear operator. Prove that T is one-to-one if and only if $\lambda = 0$ is *not* an eigenvalue of T .
2. Let $\mathbf{V} = \mathbb{R}^\infty$ be the vector space of all real sequences, and let ρ and λ be the right and left shift operators on $\mathbf{V}$,

$$\begin{aligned}\rho(a_1, a_2, a_3, \dots) &= (0, a_1, a_2, \dots), \\ \lambda(a_1, a_2, a_3, \dots) &= (a_2, a_3, a_4, \dots).\end{aligned}$$

Recall that because $\mathbf{V}$ is infinite dimensional, neither ρ nor λ have characteristic polynomials.

- (i) Determine all eigenvalues of ρ and their corresponding eigenvectors, or prove that none exist.
 - (ii) Determine all eigenvalues of λ and their corresponding eigenvectors, or prove that none exist.
3. Let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

- (i) Find the characteristic polynomial of A , and use it to determine all eigenvalues of A .
 - (ii) Determine whether A is diagonalizable. If A is not diagonalizable, explain why. If A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.
4. Let $\mathbf{V} = \mathbb{F}^2$ and let $T(x, y) = (y, x)$.
 - (i) Find the characteristic polynomial of T .
 - (ii) Find all eigenvalues and corresponding eigenvectors of T .
 5. Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{C}$. Prove that if $T \in \mathcal{L}(\mathbf{V})$, then T has eigenvectors.
 6. Let $\mathbf{V} = \mathbb{F}^2$ and let $T(x, y) = (y, -x)$.
 - (i) Prove that T does not have eigenvalues if $\mathbb{F} = \mathbb{R}$.
 - (ii) Determine all eigenvalues of T if $\mathbb{F} = \mathbb{C}$.
 - (iii) Find a basis for $\mathbf{V}$ consisting of eigenvectors of T if $\mathbb{F} = \mathbb{C}$.
 7. Let $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be given by $T(x, y) = (x + y, y)$.
 - (i) Find all eigenvalues and eigenvectors of T .
 - (ii) Determine whether T is diagonalizable. If T is not diagonalizable, explain why. If T is diagonalizable, then find a basis for $\mathbb{C}^2$ consisting of eigenvectors of T .
 8. Let $\mathbf{V} = M_{n \times n}(\mathbb{R})$ be given by $T(A) = A^t$, the transpose of A .
 - (i) Show that the only eigenvalues of T are $\lambda = -1$ and $\lambda = 1$, and describe the corresponding eigenvectors.
 - (ii) Find an ordered basis β for $M_{2 \times 2}(\mathbb{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.
 9. Let $\mathbf{V}$ be a finite dimensional vector space, and let $T \in \mathcal{L}(\mathbf{V})$. Prove that if $\text{rank}(T) = k$, then T has at most $k + 1$ distinct eigenvalues. Give an example where T has strictly fewer than $k + 1$ distinct eigenvalues, and one where T has exactly $k + 1$ distinct eigenvalues.
 10. Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{R}$, let $T \in \mathcal{L}(\mathbf{V})$, and let λ be an eigenvalue of T . Prove that there exists $\delta > 0$ such that for all scalars α such that $0 < |\alpha - \lambda| < \delta$, the linear transformation $T - \alpha I_{\mathbf{V}}$ is invertible.