Math 561 - Homework 9

Due Wednesday November 25

1. Let $T\colon \mathbf{P}_2(\mathbb{R}) \to \mathbf{P}_2(\mathbb{R})$ be given by $T(f) = 2f(x) - f'(x)$. You may take for granted that T is a linear operator. Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors, and give the Jordan canonical form of T.

2. Let U be a linear operator on a finite dimensional vector space $\mathbf{V}$.

   (i) Show that $\mathbf{N}(U) \subseteq \mathbf{N}(U^2) \subseteq \cdots \subseteq \mathbf{N}(U^k) \subseteq \mathbf{N}(U^{k+1}) \subseteq \cdots$

   (ii) Show that if $\operatorname{rank}(U^m) = \operatorname{rank}(U^{m+1})$ for some positive integer m, then we will have $\operatorname{rank}(U^m) = \operatorname{rank}(U^k)$ for all $k \geq m$.

   (iii) Show that if $\operatorname{rank}(U^m) = \operatorname{rank}(U^{m+1})$ for some positive integer m, then $\mathbf{N}(U^m) = \mathrm{N}(U^k)$ for all $k \geq m$.

3. Let T be a linear operator on a finite dimensional vector space $\mathbf{V}$ whose characteristic polynomial splits, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. Prove that T is diagonalizable if and only if $\operatorname{rank}((T - \lambda_i I)^2) = \operatorname{rank}(T - \lambda_i I)$ for $i = 1, \ldots, k$.

4. Let T be a linear operator on a finite dimensional vector space whose characteristic polynomial splits, and let λ be an eigenvalue of T.

   (i) Suppose that γ is a basis for K_λ consisting of the union of q disjoint cycles of generalized eigenvectors. Prove that $q \leq \dim(E_\lambda)$.

   (ii) Let β be a Jordan canonical basis for T, and suppose that $[T]_\beta^\beta$ has exactly q Jordan blocks that have λ in the diagonal entries. Prove that $q \leq \dim(E_\lambda)$.

5. Let T be a linear operators on a finite dimensional vector space of $\mathbf{V}$, let λ be an eigenvalue of T, and let E_λ and K_λ be the eigenspace and generalized eigenspace, respectively. Let U be an invertible linear operator on $\mathbf{V}$ such that $UT = TU$. Prove that $U(E_\lambda) = E_\lambda$ and $U(K_\lambda) = K_\lambda$.

6. Let A be the matrix
$$A = \begin{pmatrix} 0 & 0 & 8 & 0 \\ 1 & 0 & -12 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$
   Determine whether A has a Jordan canonical form, and if so, find a Jordan canonical basis and the corresponding Jordan canonical form for A.

7. Find a formula for the nth power of the matrices
$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

8. Consider the first order continuous time evolution problem
$$\frac{dx_1}{dt} = -r_1 x_1$$
$$\frac{dx_2}{dt} = r_1 x_1 - r_2 x_2.$$
   In class we disscussed the general solution when $r_1 \neq r_2$, in which case the corresponding matrix is diagonalizable.

   Consider now the case in which $r_1 = r_2$, so that the resulting matrix is not diagonalizable. Use a Jordan Canonical Form to obtain a formula for $x_1(t)$ and $x_2(t)$ at any given time t.