

Math 561 - Homework 1

SOLUTIONS

Prof. Arturo Magidin

1. Let $\mathbf{V}$ be a vector space. We saw in class that the intersection of any family of subspaces of $\mathbf{V}$ is again a subspace of $\mathbf{V}$. Give an example of a vector space $\mathbf{V}$, and two subspaces $\mathbf{W}_1$ and $\mathbf{W}_2$ of $\mathbf{V}$, such that $\mathbf{W}_1 \cup \mathbf{W}_2$ is **not** a subspace of $\mathbf{V}$. You do not need to prove that $\mathbf{W}_1$ and $\mathbf{W}_2$ are subspaces (but be sure they are), but you must explicitly show that $\mathbf{W}_1 \cup \mathbf{W}_2$ is not a subspace.

Answer. While there are many possible examples, an easy one is the following: let $\mathbf{V} = \mathbb{R}^2$ be the usual vector space over $\mathbb{R}$, and let

$$\mathbf{W}_1 = \{(a, 0) \in \mathbf{V} \mid a \in \mathbb{R}\}$$

$$\mathbf{W}_2 = \{(0, b) \in \mathbf{V} \mid b \in \mathbb{R}\}.$$

That is, $\mathbf{W}_1$ is the X -axis in the plane, and $\mathbf{W}_2$ is the Y -axis. They are both subspaces of $\mathbf{V}$. To show that $\mathbf{W}_1 \cup \mathbf{W}_2$ is not a vector space, note that $\mathbf{x} = (1, 0) \in \mathbf{W}_1 \subseteq \mathbf{W}_1 \cup \mathbf{W}_2$, and $\mathbf{y} = (0, 1) \in \mathbf{W}_2 \subseteq \mathbf{W}_1 \cup \mathbf{W}_2$. So $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbf{W}_1 \cup \mathbf{W}_2$; but $\mathbf{x} + \mathbf{y} = (1, 1)$ is in neither $\mathbf{W}_1$ nor $\mathbf{W}_2$, so it is not in $\mathbf{W}_1 \cup \mathbf{W}_2$. That is, the set $\mathbf{W}_1 \cup \mathbf{W}_2$ is not closed under vector addition, so it is not a subspace.

2. In fact, prove that if $\mathbf{W}_1$ and $\mathbf{W}_2$ are subspaces of a vector space $\mathbf{V}$, then $\mathbf{W}_1 \cup \mathbf{W}_2$ is also a subspace if and only if either $\mathbf{W}_1 \subseteq \mathbf{W}_2$, or $\mathbf{W}_2 \subseteq \mathbf{W}_1$.

Proof. If $\mathbf{W}_1 \subseteq \mathbf{W}_2$ then $\mathbf{W}_1 \cup \mathbf{W}_2 = \mathbf{W}_2$ is a subspace; if $\mathbf{W}_2 \subseteq \mathbf{W}_1$ then $\mathbf{W}_1 \cup \mathbf{W}_2 = \mathbf{W}_1$ is a subspace.

Conversely, assume that $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace. If $\mathbf{W}_1 \subseteq \mathbf{W}_2$, we are done. If $\mathbf{W}_1 \not\subseteq \mathbf{W}_2$, we need to prove that $\mathbf{W}_2 \subseteq \mathbf{W}_1$. To that end, let $\mathbf{v} \in \mathbf{W}_2$; we must prove that $\mathbf{v} \in \mathbf{W}_1$.

Let $\mathbf{x} \in \mathbf{W}_1$ with $\mathbf{x} \notin \mathbf{W}_2$. Since $\mathbf{x} \in \mathbf{W}_1$ and $\mathbf{v} \in \mathbf{W}_2$, then $\mathbf{x}, \mathbf{v} \in \mathbf{W}_1 \cup \mathbf{W}_2$; and since we are assuming that $\mathbf{W}_1 \cup \mathbf{W}_2$ is a subspace, we must have $\mathbf{x} + \mathbf{v} \in \mathbf{W}_1 \cup \mathbf{W}_2$. So $\mathbf{x} + \mathbf{v} \in \mathbf{W}_1$ or $\mathbf{x} + \mathbf{v} \in \mathbf{W}_2$.

If $\mathbf{x} + \mathbf{v} \in \mathbf{W}_2$, then since $\mathbf{v} \in \mathbf{W}_2$ and $\mathbf{W}_2$ is a subspace, we must have $\mathbf{x} = (\mathbf{x} + \mathbf{v}) - \mathbf{v} \in \mathbf{W}_2$; but we chose $\mathbf{x}$ specifically so that $\mathbf{x} \notin \mathbf{W}_2$. Thus, this case cannot occur.

We must therefore have that $\mathbf{x} + \mathbf{v} \in \mathbf{W}_1$. Since $\mathbf{x} \in \mathbf{W}_1$ and $\mathbf{W}_1$ is subspace, we must have that $\mathbf{v} = (\mathbf{x} + \mathbf{v}) - \mathbf{x} \in \mathbf{W}_1$, which is what we wanted to show.

This shows that $\mathbf{W}_2 \subseteq \mathbf{W}_1$, and we are done. $\square$

Definition. If S_1 and S_2 are two nonempty subsets of a vector space $\mathbf{V}$, then the *sum* of S_1 and S_2 , denoted $S_1 + S_2$, is the set $S_1 + S_2 = \{x + y \mid x \in S_1 \text{ and } y \in S_2\}$.

3. Show that if $\mathbf{W}_1$ and $\mathbf{W}_2$ are subspaces, then $\mathbf{W}_1 + \mathbf{W}_2$ is a subspace of $\mathbf{V}$, and in fact is the smallest subspace of $\mathbf{V}$ that contains $\mathbf{W}_1$ and $\mathbf{W}_2$.

Proof. To show that $\mathbf{W}_1 + \mathbf{W}_2$ is a subspace, note that because $\mathbf{W}_1$ and $\mathbf{W}_2$ are both subspaces, then $\mathbf{0} \in \mathbf{W}_1$ and $\mathbf{0} \in \mathbf{W}_2$; so $\mathbf{0} = \mathbf{0} + \mathbf{0} \in \mathbf{W}_1 + \mathbf{W}_2$.

Next let $\mathbf{x}, \mathbf{y} \in \mathbf{W}_1 + \mathbf{W}_2$ and $\alpha \in F$. Then there exist $\mathbf{x}_1, \mathbf{y}_1 \in \mathbf{W}_1$ and $\mathbf{x}_2, \mathbf{y}_2 \in \mathbf{W}_2$ such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$. Then

$$\begin{aligned} \mathbf{x} + \alpha \mathbf{y} &= (\mathbf{x}_1 + \mathbf{x}_2) + \alpha(\mathbf{y}_1 + \mathbf{y}_2) = \mathbf{x}_1 + \mathbf{x}_2 + \alpha \mathbf{y}_1 + \alpha \mathbf{y}_2 \\ &= (\mathbf{x}_1 + \alpha \mathbf{y}_1) + (\mathbf{x}_2 + \alpha \mathbf{y}_2). \end{aligned}$$

Since each of $\mathbf{W}_1$ and $\mathbf{W}_2$ are subspaces and $\mathbf{x}_i, \mathbf{y}_i \in \mathbf{W}_i$, it follows that $\mathbf{x}_1 + \alpha \mathbf{y}_1 \in \mathbf{W}_1$ and $\mathbf{x}_2 + \alpha \mathbf{y}_2 \in \mathbf{W}_2$. Thus, $\mathbf{x} + \alpha \mathbf{y}$ can be expressed as the sum of a vector in $\mathbf{W}_1$ and a vector in $\mathbf{W}_2$, proving that it lies in $\mathbf{W}_1 + \mathbf{W}_2$. This proves that $\mathbf{W}_1 + \mathbf{W}_2$ is a subspace of $\mathbf{V}$.

Next we prove that $\mathbf{W}_1 + \mathbf{W}_2$ contains both $\mathbf{W}_1$ and $\mathbf{W}_2$. Indeed, let $\mathbf{x} \in \mathbf{W}_1$. Then we can write $\mathbf{x} = \mathbf{x} + \mathbf{0}$, showing that $\mathbf{x}$ is the sum of a vector in $\mathbf{W}_1$ and a vector in $\mathbf{W}_2$, which proves that it lies in $\mathbf{W}_1 + \mathbf{W}_2$. Symmetrically, if $\mathbf{y} \in \mathbf{W}_2$, then writing $\mathbf{y} = \mathbf{0} + \mathbf{y}$ shows that $\mathbf{W}_2 \subseteq \mathbf{W}_1 + \mathbf{W}_2$.

Finally, to prove that $\mathbf{W}_1 + \mathbf{W}_2$ is the smallest subspace of $\mathbf{V}$ that contains both $\mathbf{W}_1$ and $\mathbf{W}_2$, let $\mathbf{W}$ be *any* subspace of $\mathbf{V}$ that contains them. Let $\mathbf{v} \in \mathbf{W}_1 + \mathbf{W}_2$; we must show that $\mathbf{v} \in \mathbf{W}$. Since $\mathbf{v} \in \mathbf{W}_1 + \mathbf{W}_2$, we can write $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ with $\mathbf{w}_1 \in \mathbf{W}_1$ and $\mathbf{w}_2 \in \mathbf{W}_2$. Then $\mathbf{w}_1 \in \mathbf{W}_1 \subseteq \mathbf{W}$ and $\mathbf{w}_2 \in \mathbf{W}_2 \subseteq \mathbf{W}$; and because $\mathbf{W}$ is a subspace, it is closed under vector addition, so we must have $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 \in \mathbf{W}$. This proves that $\mathbf{W}_1 + \mathbf{W}_2 \subseteq \mathbf{W}$, which is what we needed to prove. $\square$

4. Show that if S_1 and S_2 are arbitrary subsets of a vector space $\mathbf{V}$, then

$$\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2).$$

Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal, and one in which they are unequal.

Proof. Since $S_1 \cap S_2 \subseteq S_1$, then $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1)$. Symmetrically, we also have $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_2)$. And since it is contained in both spans, then we have

$$\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2).$$

For an easy example where they are equal, take $S_1 = S_2$. For a less trivial example, consider $\mathbf{V} = \mathbb{R}^3$, $S_1 = \{(1, 0, 0), (0, 1, 0)\}$, and $S_2 = \{(1, 0, 0), (0, 0, 1)\}$, so $S_1 \cap S_2 = \{(1, 0, 0)\}$. Then $\text{span}(S_1)$ is the XY -plane; $\text{span}(S_2)$ is the XZ -plane; so $\text{span}(S_1) \cap \text{span}(S_2)$ is the X -axis, which is the same as $\text{span}(S_1 \cap S_2)$.

For an example where they are unequal, consider $\mathbf{V} = \mathbb{R}^2$, $S_1 = \{(1, 0)\}$ and $S_2 = \{(2, 0)\}$. Then $S_1 \cap S_2 = \emptyset$, so $\text{span}(S_1 \cap S_2) = \{(0, 0)\}$. But both $\text{span}(S_1)$ and $\text{span}(S_2)$ are the X -axis, and hence so is $\text{span}(S_1) \cap \text{span}(S_2)$. $\square$

5. Show that if S_1 and S_2 are subsets of a vector space $\mathbf{V}$, then

$$\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2).$$

Proof. Note that since $S_1 \subseteq S_1 \cup S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_1 \cup S_2)$. Symmetrically, we also have $\text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$. And since $\text{span}(S_1 \cup S_2)$ is a subspace that contains both $\text{span}(S_1)$ and $\text{span}(S_2)$, then by Problem 3 it follows that

$$\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2).$$

For the converse inclusion, note that $S_1 \subseteq \text{span}(S_1) \subseteq \text{span}(S_1) + \text{span}(S_2)$, and likewise that $S_2 \subseteq \text{span}(S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$. Therefore,

$$S_1 \cup S_2 \subseteq \text{span}(S_1) + \text{span}(S_2).$$

Since $\text{span}(S_1) + \text{span}(S_2)$ is a subspace (by Problem 3) that contains $S_1 \cup S_2$, and $\text{span}(S_1 \cup S_2)$ is the smallest subspace that contains $S_1 \cup S_2$, then we must have

$$\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2).$$

This proves the equality. $\square$

6. Show that for any subset S of $\mathbf{V}$, $\text{span}(\text{span}(S)) = \text{span}(S)$.

Proof. We know that the span of a subset of $\mathbf{V}$ always contains the subset, so we must have

$$\text{span}(S) \subseteq \text{span}(\text{span}(S)).$$

On the other hand, since $\text{span}(S) \subseteq \text{span}(S)$, and $\text{span}(S)$ is a subspace, then it must contain the span of the set $\text{span}(S)$; that is,

$$\text{span}(\text{span}(S)) \subseteq \text{span}(S).$$

This proves the equality. $\square$

7. Let S be a finite subset of $\mathbf{V}$ with exactly k elements. Prove that S is linearly independent if and only if $\dim(\text{span}(S)) = k$.

Proof. Note that S always generates $\text{span}(S)$.

If S is linearly independent, then it is a linearly independent subset of $\text{span}(S)$ that generates, so it is a basis for $\text{span}(S)$, and therefore, $\dim(\text{span}(S)) = |S| = k$.

Conversely, since S generates $\text{span}(S)$, then it contains a basis for $\text{span}(S)$; and since

$$\dim(\text{span}(S)) = k = |S|,$$

then this basis consists of all of S ; thus, S is a basis for $\text{span}(S)$ and therefore is linearly independent. $\square$

8. Let $\mathbf{V}$ be a vector space. Prove that a subset S of $\mathbf{V}$ is linearly independent if and only if there exists a subspace $\mathbf{W}$ of $\mathbf{V}$ such that S is a basis for $\mathbf{W}$.

Proof. If there is a subspace $\mathbf{W}$ of $\mathbf{V}$ for which S is a basis, then S must be linearly independent.

Conversely, if S is linearly independent, then let $\mathbf{W} = \text{span}(S)$. Then S is a linearly independent subset of $\mathbf{W}$ that generates $\mathbf{W}$, hence is a basis for $\mathbf{W}$. $\square$

9. We proved the following statements are true in the case of finite dimensional vector spaces. Use $\mathbf{V} = \mathbb{R}[x]$, the vector space of all polynomials with real coefficients (vector space over $\mathbb{R}$, of dimension $\aleph_0$) to give explicit counterexamples to each of them. Here, $|X|$ means the cardinality of the set X .

- (i) If $S \subseteq \mathbf{V}$ is a generating set with $|S| = \dim(\mathbf{V})$, then S is a basis for $\mathbf{V}$.

Answer. Recall that a basis for $\mathbf{V}$ is given by

$$\beta = \{1, x, x^2, \dots, x^n, \dots\}.$$

Since β spans, adding any vector that is not already in β to β will yield a linearly dependent set; and adding just one vector will yield a set that also has $\aleph_0$ vectors in it. So take, for example, $S = \beta \cup \{0\}$ (or $S = \beta \cup \{1+x\}$, or any vector not already in S) to get a generating set with $|S| = \dim(\mathbf{V})$ that is not linearly independent and therefore not a basis.

- (ii) If $L \subseteq \mathbf{V}$ is a linearly independent set with $|L| = \dim(\mathbf{V})$, then L is a basis for $\mathbf{V}$.

Answer. Since β is linearly independent, if $\mathbf{v} \in \beta$, then $\mathbf{v} \notin \text{span}(\beta - \{\mathbf{v}\})$. So we can take $S = \{x, x^2, \dots, x^n, \dots\}$. Then S is not a basis for $\mathbf{V}$, since it does not generate; but $|S| = \aleph_0 = \dim(\mathbf{V})$.

- (iii) If $\mathbf{W}$ is a subspace of $\mathbf{V}$, and $\dim(\mathbf{W}) = \dim(\mathbf{V})$, then $\mathbf{W} = \mathbf{V}$.

Answer. Take $\mathbf{W} = \{p(x) \in \mathbf{V} \mid p(0) = 0\}$; that is, $\mathbf{W} = \text{span}(x, x^2, \dots, x^n, \dots)$. Then $\mathbf{W} \neq \mathbf{V}$, but $\dim(\mathbf{W}) = \aleph_0 = \dim(\mathbf{V})$.