

Math 561 - Homework 2

SOLUTIONS

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1. Let $\mathbf{V}$ be a vector space, $\mathbf{V} \neq \{\mathbf{0}\}$. Prove that $\dim(\mathbf{V}) = 1$ if and only if every nonzero vector in $\mathbf{V}$ is a basis for $\mathbf{V}$.

Proof. If $\dim(\mathbf{V}) = 1$, then any one element linearly independent subset of $\mathbf{V}$ is a basis; since $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$, it follows that every nonzero vector in $\mathbf{V}$ is a basis for $\mathbf{V}$.

Conversely, let $\mathbf{v} \neq \mathbf{0}$ in $\mathbf{V}$ (it exists because we are assuming that $\mathbf{V} \neq \{\mathbf{0}\}$). Then, by assumption, $\{\mathbf{v}\}$ is a basis for $\mathbf{V}$, so $\dim(\mathbf{V})$ is the number of vectors in that basis, namely 1. So $\dim(\mathbf{V}) = 1$. $\square$

2. Recall that if $\mathbf{V} = \{a \in \mathbb{R} \mid a > 0\}$, then $\mathbf{V}$ is a real vector space using the operations $\oplus$ and $\odot$, defined as:

$$\begin{aligned} r \oplus s &= rs \text{ for all } r, s \in \mathbf{V}, \\ \alpha \odot r &= r^\alpha \text{ for all } r \in \mathbf{V}, \alpha \in \mathbb{R}. \end{aligned}$$

Prove that if $r \in \mathbf{V}$, then $\{r\}$ is a basis for $\mathbf{V}$ if and only if $r \neq 1$. Conclude that $\dim(\mathbf{V}) = 1$.

Proof. Let r be a positive real, $r \neq 1$. To show that r spans $\mathbf{V}$, let $a \in \mathbf{V}$. Setting $\alpha = \log_r(a) = \frac{\ln(a)}{\ln(r)}$, we have that $\alpha \odot r = r^\alpha = r^{\log_r(a)} = a$. Thus, every vector in $\mathbf{V}$ is a scalar multiple of r , showing that r spans $\mathbf{V}$. Since $r \neq \mathbf{0}$, $\{r\}$ is linearly independent; thus, it is a basis.

Conversely, if $r = 1$, then $r = \mathbf{0}$, so $\{r\}$ is not linearly independent and therefore not a basis. $\square$

3. Let X be a set. Recall that $\mathbb{R}^X$, also denoted $\mathcal{F}(X, \mathbb{R})$, is the vector space of all functions $f: X \rightarrow \mathbb{R}$ under pointwise addition and scalar multiplication. That is, if $f, g \in \mathcal{F}(X, \mathbb{R})$ and $a \in \mathbb{R}$, then we define $f + g$ to be the function given by $(f + g)(x) = f(x) + g(x)$, where the addition on the right hand side is the usual addition of real numbers; and $a \cdot f$ is the function defined by $(a \cdot f)(x) = a f(x)$, where the operation on the right hand side is the usual multiplication of real numbers.

Define $\mathbb{R}^{(X)}$ be the the subset of $\mathcal{F}(X, \mathbb{R})$ given by

$$\mathbb{R}^{(X)} = \{f: X \rightarrow \mathbb{R} \mid f(x) = 0 \text{ for almost all } x \in X\},$$

where “almost all” means “all except perhaps for finitely many.”

- (i) Prove that $\mathbb{R}^{(X)}$ is a subspace of $\mathcal{F}(X, \mathbb{R})$.

Proof. Note that the zero vector is the constant function $f(x) = 0$ for all $x \in X$, which lies in $\mathbb{R}^{(X)}$. Now suppose that $f, g \in \mathbb{R}^{(X)}$ and $\alpha \in \mathbb{R}$. Since $f \in \mathbb{R}^{(X)}$, there exists $X_f \subseteq X$, finite, such that $f(x) = 0$ for all $x \notin X_f$; similarly, there exists $X_g \subseteq X$ finite such that $g(x) = 0$ for all $x \notin X_g$. Letting $Y = X_f \cup X_g$, we have that Y is a finite subset of X with the property that if $y \notin Y$, then $f(y) = g(y) = 0$. Therefore, for all $y \notin Y$, we have

$$(f + \alpha g)(y) = f(y) + \alpha g(y) = 0 + \alpha 0 = 0.$$

That is, $(f + \alpha g)(x) = 0$ for all x outside the finite set Y , and thus $(f + \alpha g)(x) = 0$ for almost all $x \in X$. This proves that $\mathbb{R}^{(X)}$ is a subspace of $\mathcal{F}(X, \mathbb{R})$. $\square$

- (ii) Prove that $\dim(\mathbb{R}^{(X)}) = |X|$, the cardinality of the set X . HINT: Construct a basis that is indexed by elements of X .

Proof. For each $x \in X$, let $\delta_x \in \mathbb{R}^{(X)}$ be the indicator function, which is defined by

$$\delta_x(y) = \begin{cases} 0 & \text{if } y \neq x, \\ 1 & \text{if } y = x. \end{cases}$$

for all $y \in X$. That is, the function δ_x takes value 1 at x and 0 elsewhere. Since $\delta_x: X \rightarrow \mathbb{R}$ takes nonzero values at only a finite subset of X (namely $\{x\}$), it is indeed an element of $\mathbb{R}^{(X)}$.

We claim that $B = \{\delta_x \mid x \in X\}$ is a basis for $\mathbb{R}^{(X)}$.

First, to show that it spans, let $f \in \mathbb{R}^{(X)}$; then there exists a finite subset $X_0 \subseteq X$ such that $f(y) = 0$ for all $y \notin X_0$. Then we have that

$$f = \sum_{x \in X_0} f(x)\delta_x.$$

Indeed, if $y \notin x$, then the right hand side evaluates to 0 (since $y \neq x$ for all $x \in X_0$), which is the same as $f(y)$; and if $x \in X_0$, then the right hand side evaluates to $f(x)\delta_x(x) = f(x)$. Thus, the right hand side take the same values of f everywhere, giving equality.

Second, to prove that B is linearly independent, let $\delta_{x_1}, \dots, \delta_{x_n}$ be a finite collection of pairwise distinct elements of B , and suppose that $a_1, \dots, a_n \in \mathbb{R}$ are such that

$$a_1\delta_{x_1} + \dots + a_n\delta_{x_n} = \mathbf{0}.$$

We want to show that $a_1 = \dots = a_n$. Plugging in x_i into both sides for $i = 1, \dots, n$, we get $a_i = 0$, proving that $a_1 = \dots = a_n$.

Thus, B is a basis for $\mathbb{R}^{(X)}$. This proves that $\dim(\mathbb{R}^{(X)}) = |B| = |X|$, as desired. $\square$

4. Let $\mathbf{V}$ be a vector space, and let $\mathbf{W}_1$ and $\mathbf{W}_2$ be finite dimensional subspaces of $\mathbf{V}$. Prove that

$$\dim(\mathbf{W}_1 \cap \mathbf{W}_2) + \dim(\mathbf{W}_1 + \mathbf{W}_2) = \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2).$$

HINT: Start with a basis for $\mathbf{W}_1 \cap \mathbf{W}_2$, and extend it to a basis for $\mathbf{W}_1$ and to a basis for $\mathbf{W}_2$.

Proof. Suppose that $\dim(\mathbf{W}_1 \cap \mathbf{W}_2) = k$, $\dim(\mathbf{W}_1) = n$, and $\dim(\mathbf{W}_2) = m$. We need to show that $\dim(\mathbf{W}_1 + \mathbf{W}_2) = (n + m) - k$. Note that $n \geq k$ and $m \geq k$, because $\mathbf{W}_1 \cap \mathbf{W}_2$ is a subspace of both $\mathbf{W}_1$ and of $\mathbf{W}_2$.

To that end, let $B_0 = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be a basis for $\mathbf{W}_1 \cap \mathbf{W}_2$. This is a linearly independent subset of $\mathbf{W}_1$, so we can extend it to a basis for $\mathbf{W}_1$ by adding vectors $B_1 = \{\mathbf{x}_1, \dots, \mathbf{x}_{n-k}\}$; that is, $B_0 \cap B_1 = \emptyset$ and $B_0 \cup B_1$ is a basis for $\mathbf{W}_1$. Symmetrically, there exists a set $B_2 = \{\mathbf{y}_1, \dots, \mathbf{y}_{m-k}\}$ that is disjoint from B_0 and such that $B_0 \cup B_2$ is a basis for $\mathbf{W}_2$.

We claim that $B = B_0 \cup B_1 \cup B_2$ is a basis for $\mathbf{W}_1 + \mathbf{W}_2$; this will prove the required equality: note that B_0 , B_1 , and B_2 are pairwise disjoint (since we are choosing B_2 linearly independent from B_0 , so the elements of B_2 cannot lie in $\mathbf{W}_1$, hence cannot also be elements of B_1), and therefore

$$|B_0 \cup B_1 \cup B_2| = k + (n - k) + (m - k) = (n + m) - k.$$

Since $B_0 \cup B_1 \subseteq B$, we know that

$$\mathbf{W}_1 = \text{span}(B_0 \cup B_1) \subseteq \text{span}(B).$$

Symmetrically, $\mathbf{W}_2 \subseteq \text{span}(B)$. Since $\mathbf{W}_1 + \mathbf{W}_2$ is the smallest subspace of $\mathbf{V}$ that contains both $\mathbf{W}_1$ and $\mathbf{W}_2$, this implies that $\mathbf{W}_1 + \mathbf{W}_2 \subseteq \text{span}(B)$. Conversely, note that $B \subseteq \mathbf{W}_1 \cup \mathbf{W}_2$ and $\mathbf{W}_1 \cup \mathbf{W}_2 \subseteq \mathbf{W}_1 + \mathbf{W}_2$, so $\text{span}(B) \subseteq \mathbf{W}_1 + \mathbf{W}_2$. This proves that B spans $\mathbf{W}_1 + \mathbf{W}_2$.

So we just need to prove that B is linearly independent. Suppose that we have scalars

$$a_1, \dots, a_k, b_1, \dots, b_{n-k}, c_1, \dots, c_{m-k}$$

such that

$$\mathbf{0} = a_1\mathbf{w}_1 + \dots + a_k\mathbf{w}_k + b_1\mathbf{x}_1 + \dots + b_{n-k}\mathbf{x}_{n-k} + c_1\mathbf{y}_1 + \dots + c_{m-k}\mathbf{y}_{m-k}. \quad (1)$$

Then

$$a_1 \mathbf{w}_1 + \cdots + a_k \mathbf{w}_k + b_1 \mathbf{x}_1 + \cdots + b_{n-k} \mathbf{x}_{n-k} = -c_1 \mathbf{y}_1 - \cdots - c_{m-k} \mathbf{y}_{m-k}.$$

The left hand side is a linear combination of vectors in B_1 , hence it lies in $\mathbf{W}_1$; and the right hand side is a linear combination of vectors in B_2 , so it lies in $\mathbf{W}_2$. Thus, this vector lies in $\mathbf{W}_1 \cap \mathbf{W}_2$. Therefore, there exist $d_1, \dots, d_k$ such that

$$d_1 \mathbf{w}_1 + \cdots + d_k \mathbf{w}_k = -c_1 \mathbf{y}_1 - \cdots - c_{m-k} \mathbf{y}_{m-k},$$

or equivalently that

$$d_1 \mathbf{w}_1 + \cdots + d_k \mathbf{w}_k + c_1 \mathbf{y}_1 + \cdots + c_{m-k} \mathbf{y}_{m-k} = \mathbf{0}.$$

Since this is a linear combination of vectors in B_2 , which is linearly independent, it follows that $d_1 = \cdots = d_k = c_1 = \cdots = c_{m-k} = 0$. In particular, $c_1 = \cdots = c_{m-k} = 0$.

Plugging this into (1), we get

$$a_1 \mathbf{w}_1 + \cdots + a_k \mathbf{w}_k + b_1 \mathbf{x}_1 + \cdots + b_{n-k} \mathbf{x}_{n-k} = \mathbf{0},$$

and since this is a linear combination of vectors in B_1 , which is linearly independent, we conclude that $a_1 = \cdots = a_k = b_1 = \cdots = b_{n-k}$.

Thus, B is linearly independent, and hence a basis for $\mathbf{W}_1 + \mathbf{W}_2$. Thus we have

$$\begin{aligned} \dim(\mathbf{W}_1 \cap \mathbf{W}_2) + \dim(\mathbf{W}_1 + \mathbf{W}_2) &= |B_0| + |B| \\ &= k + (n + m - k) \\ &= n + m \\ &= |B_1| + |B_2| \\ &= \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2). \quad \square \end{aligned}$$

5. Let $\mathbf{V}$ be a finite dimensional vector space, and let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\mathbf{V}$. Let α be a scalar, and i, j be integers, $1 \leq i, j \leq n$, $i \neq j$. Prove that if you replace $\mathbf{v}_i$ with $\mathbf{v}_i + \alpha \mathbf{v}_j$ in β , the resulting set is still a basis for $\mathbf{V}$.

Proof. Let γ be the result of replacing $\mathbf{v}_i$ with $\mathbf{v}_i + \alpha \mathbf{v}_j$ in β .

First, note that $\mathbf{v}_k \in \gamma$ for $k \neq i$. And since

$$\mathbf{v}_i = (\mathbf{v}_i + \alpha \mathbf{v}_j) - \alpha \mathbf{v}_j \in \text{span}(\gamma),$$

we have that all elements of β lie in the span of γ . Thus, $\mathbf{V} = \text{span}(\beta) \subseteq \text{span}(\gamma) \subseteq \mathbf{V}$, so γ spans $\mathbf{V}$.

Since γ is a spanning set with n elements in an n -dimensional vector space, it follows that γ is a basis.

Alternatively, we can directly prove that γ is linearly independent: suppose that

$$\mathbf{0} = a_1 \mathbf{v}_1 + \cdots + a_{i-1} \mathbf{v}_{i-1} + a_i (\mathbf{v}_i + \alpha \mathbf{v}_j) + a_{i+1} \mathbf{v}_{i+1} + \cdots + a_n \mathbf{v}_n.$$

We want to prove that $a_1 = \cdots = a_n = 0$. We have

$$\begin{aligned} 0 &= a_1 \mathbf{v}_1 + \cdots + a_{i-1} \mathbf{v}_{i-1} + a_i (\mathbf{v}_i + \alpha \mathbf{v}_j) + a_{i+1} \mathbf{v}_{i+1} + \cdots + a_n \mathbf{v}_n \\ &= a_1 \mathbf{v}_1 + \cdots + a_{i-1} \mathbf{v}_{i-1} + a_i \mathbf{v}_i + \cdots + (a_j + \alpha a_i) \mathbf{v}_j + \cdots + a_n \mathbf{v}_n. \end{aligned}$$

Since this is a linear combination of vectors in β , we conclude that

$$a_1 = \cdots = a_i = \cdots = a_{j-1} = a_{j+1} = \cdots = a_n = 0,$$

and $a_j + \alpha a_i = 0$. But since $a_i = 0$, this also gives $a_j = 0$, proving that γ is linearly independent, as claimed. $\square$

6. Does the equality in Problem 4 hold if we do not assume that $\mathbf{W}_1$ and $\mathbf{W}_2$ are finite dimensional? Prove or given a counterexample. SUGGESTION: Recall that if κ and λ are cardinals, and at least one of them is infinite, then $\kappa + \lambda = \max(\kappa, \lambda)$.

Answer. Yes, the equality holds.

If both $\mathbf{W}_1$ and $\mathbf{W}_2$ are finite dimensional, then we have proven the result already. If at least one of them is infinite dimensional, say that $\dim(\mathbf{W}_1) \leq \dim(\mathbf{W}_2)$ with $\dim(\mathbf{W}_2)$ infinite. Then $\dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) = \max\{\dim(\mathbf{W}_1), \dim(\mathbf{W}_2)\} = \dim(\mathbf{W}_2)$.

On the other hand, since $\mathbf{W}_1 \cap \mathbf{W}_2$ is a subspace of $\mathbf{W}_1$, then $\dim(\mathbf{W}_1 \cap \mathbf{W}_2) \leq \dim(\mathbf{W}_1) \leq \dim(\mathbf{W}_2)$. If β is a basis for $\mathbf{W}_1$ and γ is a basis for $\mathbf{W}_2$, then $\beta \cup \gamma$ is a generating set for $\mathbf{W}_1 + \mathbf{W}_2$, hence it contains a basis for $\mathbf{W}_1 + \mathbf{W}_2$. So

$$\dim(\mathbf{W}_1 + \mathbf{W}_2) \leq |\beta \cup \gamma| \leq |\beta| + |\gamma| = \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) = \dim(\mathbf{W}_2).$$

But $\mathbf{W}_2$ is a subspace of $\mathbf{W}_1 + \mathbf{W}_2$, so $\dim(\mathbf{W}_2) \leq \dim(\mathbf{W}_1 + \mathbf{W}_2)$. This proves that under these assumptions, we have $\dim(\mathbf{W}_1 + \mathbf{W}_2) = \dim(\mathbf{W}_2)$. In particular, $\dim(\mathbf{W}_1 + \mathbf{W}_2)$ is infinite and greater than or equal to $\dim(\mathbf{W}_1 \cap \mathbf{W}_2)$.

So $\dim(\mathbf{W}_1 \cap \mathbf{W}_2) + \dim(\mathbf{W}_1 + \mathbf{W}_2) = \dim(\mathbf{W}_1 + \mathbf{W}_2) = \dim(\mathbf{W}_2)$. Thus, we have:

$$\dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) = \dim(\mathbf{W}_2) = \dim(\mathbf{W}_1 + \mathbf{W}_2) = \dim(\mathbf{W}_1 \cap \mathbf{W}_2) + \dim(\mathbf{W}_1 + \mathbf{W}_2),$$

which proves that the equality also holds in this case. $\square$

NOTE: In fact, we can prove the equality in the same manner as we did in Problem 4, by starting with a basis for $\mathbf{W}_1 \cap \mathbf{W}_2$, extending it to bases for each of $\mathbf{W}_1$ and of $\mathbf{W}_2$, and then showing the union is a basis for $\mathbf{W}_1 + \mathbf{W}_2$; the calculation of the cardinalities of the bases will yield the equality as well.