

Math 561 - Homework 3

SOLUTIONS

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1. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over $\mathbb{F}$. Let $\mathbf{V} \times \mathbf{W}$ be the set

$$\mathbf{V} \times \mathbf{W} = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in \mathbf{V}, \mathbf{w} \in \mathbf{W}\},$$

and define addition and scalar multiplication coordinate-wise; that is,

$$\begin{aligned}(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) &= (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2), \\ \alpha(\mathbf{v}, \mathbf{w}) &= (\alpha\mathbf{v}, \alpha\mathbf{w}).\end{aligned}$$

This makes $\mathbf{V} \times \mathbf{W}$ into a vector space. (You may take this for granted, but if you have not seen this proven before I encourage you to prove it for yourself). This vector space is called the (DIRECT) PRODUCT OF $\mathbf{V}$ AND $\mathbf{W}$.

- (i) Prove that if $\beta = \{\mathbf{v}_i\}_{i \in I}$ and $\gamma = \{\mathbf{w}_j\}_{j \in J}$ are bases for $\mathbf{V}$ and $\mathbf{W}$, respectively, then

$$\{(\mathbf{v}_i, \mathbf{0}_\mathbf{W})\}_{i \in I} \cup \{(\mathbf{0}_\mathbf{V}, \mathbf{w}_j)\}_{j \in J}$$

is a basis for $\mathbf{V} \times \mathbf{W}$.

Proof. First, we show that the set is linearly independent. Suppose we have a finite collection of pairwise distinct vectors,

$$(\mathbf{v}_1, \mathbf{0}_\mathbf{W}), \dots, (\mathbf{v}_m, \mathbf{0}_\mathbf{W}), (\mathbf{0}_\mathbf{V}, \mathbf{w}_1), \dots, (\mathbf{0}_\mathbf{V}, \mathbf{w}_n)$$

and scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$a_1(\mathbf{v}_1, \mathbf{0}_\mathbf{W}) + \dots + a_m(\mathbf{v}_m, \mathbf{0}_\mathbf{W}) + b_1(\mathbf{0}_\mathbf{V}, \mathbf{w}_1) + \dots + b_n(\mathbf{0}_\mathbf{V}, \mathbf{w}_n) = (\mathbf{0}_\mathbf{V}, \mathbf{0}_\mathbf{W}).$$

We want to show that $a_1 = \dots = a_m = b_1 = \dots = b_n = 0$.

Performing the operations on the left hand side, we obtain

$$(a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m, b_1\mathbf{w}_1 + \dots + b_n\mathbf{w}_n) = (\mathbf{0}_\mathbf{V}, \mathbf{0}_\mathbf{W}).$$

This means that

$$\begin{aligned}a_1\mathbf{v}_1 + \dots + a_m\mathbf{v}_m &= \mathbf{0}_\mathbf{V}, \\ b_1\mathbf{w}_1 + \dots + b_n\mathbf{w}_n &= \mathbf{0}_\mathbf{W}.\end{aligned}$$

These are linear combinations of pairwise distinct vectors from β and γ , respectively, equal to $\mathbf{0}$; since these sets are linearly independent, the first equality implies $a_1 = \dots = a_m = 0$, and the second yields $b_1 = \dots = b_n = 0$, as desired.

To prove that the set spans $\mathbf{V} \times \mathbf{W}$, let $\mathbf{v} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{W}$ be arbitrary vectors. We need to express $(\mathbf{v}, \mathbf{w})$ as a linear combination of vectors in our set. Indeed, since β spans $\mathbf{V}$ and γ spans $\mathbf{W}$, we can find vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \beta$ and $\mathbf{w}_1, \dots, \mathbf{w}_\ell \in \gamma$, together with scalars $c_1, \dots, c_k$ and $d_1, \dots, d_\ell$ such that

$$\begin{aligned}\mathbf{v} &= c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k, \\ \mathbf{w} &= d_1\mathbf{w}_1 + \dots + d_\ell\mathbf{w}_\ell.\end{aligned}$$

Therefore,

$$(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^k c_i(\mathbf{v}_i, \mathbf{0}_\mathbf{W}) + \sum_{j=1}^{\ell} d_j(\mathbf{0}_\mathbf{V}, \mathbf{w}_j).$$

This shows that the set spans $\mathbf{V} \times \mathbf{W}$, and therefore that it is indeed a basis, as claimed. $\square$

- (ii) Express $\dim(\mathbf{V} \times \mathbf{W})$ in terms of $\dim(\mathbf{V})$ and $\dim(\mathbf{W})$.

Answer. From the above, we see that $\dim(\mathbf{V} \times \mathbf{W}) = |\beta| + |\gamma| = \dim(\mathbf{V}) + \dim(\mathbf{W})$. $\square$

2. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over $\mathbb{F}$, and let $\mathbf{V} \times \mathbf{W}$ be the direct product of $\mathbf{V}$ and $\mathbf{W}$. Let $\pi_1: \mathbf{V} \times \mathbf{W} \rightarrow \mathbf{V}$ be the function $\pi_1(\mathbf{v}, \mathbf{w}) = \mathbf{v}$, and $\pi_2: \mathbf{V} \times \mathbf{W} \rightarrow \mathbf{W}$ be the function $\pi_2(\mathbf{v}, \mathbf{w}) = \mathbf{w}$.

- (i) Show that π_1 and π_2 are linear transformations.

Proof. Let $(\mathbf{v}_1, \mathbf{w}_1)$ and $(\mathbf{v}_2, \mathbf{w}_2)$ be vectors in $\mathbf{V} \times \mathbf{W}$ and α a scalar. We have:

$$\begin{aligned}\pi_1((\mathbf{v}_1, \mathbf{w}_1) + \alpha(\mathbf{v}_2, \mathbf{w}_2)) &= \pi_1(\mathbf{v}_1 + \alpha\mathbf{v}_2, \mathbf{w}_1 + \alpha\mathbf{w}_2) \\ &= \mathbf{v}_1 + \alpha\mathbf{v}_2 = \pi_1(\mathbf{v}_1, \mathbf{w}_1) + \alpha\pi_1(\mathbf{v}_2, \mathbf{w}_2); \\ \pi_2((\mathbf{v}_1, \mathbf{w}_1) + \alpha(\mathbf{v}_2, \mathbf{w}_2)) &= \pi_2(\mathbf{v}_1 + \alpha\mathbf{v}_2, \mathbf{w}_1 + \alpha\mathbf{w}_2) \\ &= \mathbf{w}_1 + \alpha\mathbf{w}_2 = \pi_2(\mathbf{v}_1, \mathbf{w}_1) + \alpha\pi_2(\mathbf{v}_2, \mathbf{w}_2).\end{aligned}$$

This shows that π_1 and π_2 are both linear transformations. $\square$

- (ii) Show that given any vector space $\mathbf{Z}$ over $\mathbb{F}$ and linear transformations $T_1: \mathbf{Z} \rightarrow \mathbf{V}$ and $T_2: \mathbf{Z} \rightarrow \mathbf{W}$, there exists a unique linear transformation $U: \mathbf{Z} \rightarrow \mathbf{V} \times \mathbf{W}$ such that $\pi_1 \circ U = T_1$ and $\pi_2 \circ U = T_2$.

Proof. Suppose that $\mathbf{Z}$ is a vector space, and that we have linear transformations $T_1: \mathbf{Z} \rightarrow \mathbf{V}$ and $T_2: \mathbf{Z} \rightarrow \mathbf{W}$.

EXISTENCE OF U : Define $U: \mathbf{Z} \rightarrow \mathbf{V} \times \mathbf{W}$ by $U(\mathbf{z}) = (T_1(\mathbf{z}), T_2(\mathbf{z}))$. To verify that this is a linear transformation, let $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{Z}$, $\alpha \in \mathbb{F}$. Then

$$\begin{aligned}U(\mathbf{z}_1 + \alpha\mathbf{z}_2) &= (T_1(\mathbf{z}_1 + \alpha\mathbf{z}_2), T_2(\mathbf{z}_1 + \alpha\mathbf{z}_2)) \\ &= (T_1(\mathbf{z}_1) + \alpha T_1(\mathbf{z}_2), T_2(\mathbf{z}_1) + \alpha T_2(\mathbf{z}_2)) \quad (\text{since } T_i \text{ is linear}) \\ &= (T_1(\mathbf{z}_1), T_2(\mathbf{z}_1)) + \alpha(T_1(\mathbf{z}_2), T_2(\mathbf{z}_2)) \\ &= U(\mathbf{z}_1) + \alpha U(\mathbf{z}_2).\end{aligned}$$

By construction, we have $\pi_j \circ U(\mathbf{z}) = \pi_j(T_1(\mathbf{z}), T_2(\mathbf{z})) = T_j(\mathbf{z})$ for $j = 1, 2$; so $\pi_j \circ U = T_j$, as desired.

UNIQUENESS OF U : Suppose that $U': \mathbf{Z} \rightarrow \mathbf{V} \times \mathbf{W}$ is a linear transformation such that $\pi_1 \circ U' = T_1$ and $\pi_2 \circ U' = T_2$. We need to show that $U = U'$. To that end, let $\mathbf{z} \in \mathbf{Z}$, and let $\mathbf{v} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{W}$ be such that $U'(\mathbf{z}) = (\mathbf{v}, \mathbf{w})$. Then

$$\mathbf{v} = \pi_1(\mathbf{v}, \mathbf{w}) = \pi_1 \circ U'(\mathbf{z}) = T_1(\mathbf{z}),$$

and symmetrically $\mathbf{w} = \pi_2(\mathbf{v}, \mathbf{w}) = \pi_2 \circ U'(\mathbf{z}) = T_2(\mathbf{z})$. So $U'(\mathbf{z}) = (T_1(\mathbf{z}), T_2(\mathbf{z})) = U(\mathbf{z})$. As this holds for every $\mathbf{z} \in \mathbf{Z}$, we conclude that $U = U'$, proving uniqueness. $\square$

3. Let $\mathbf{V} = \mathbb{R}^\infty$ be the vector space of real sequences with pointwise addition and scalar multiplication. Define $\rho, \lambda: \mathbf{V} \rightarrow \mathbf{V}$ to be the “right shift” and “left shift” functions:

$$\begin{aligned}\rho(a_1, a_2, \dots) &= (0, a_1, a_2, \dots) \\ \lambda(a_1, a_2, \dots) &= (a_2, a_3, \dots).\end{aligned}$$

- (i) Show that ρ and λ are linear transformations.

Proof. Let $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$ be vectors in $\mathbf{V}$, and let $\alpha \in \mathbb{R}$. We have:

$$\begin{aligned}\lambda(\mathbf{a} + \alpha\mathbf{b}) &= \lambda(a_1 + \alpha b_1, a_2 + \alpha b_2, a_3 + \alpha b_3, \dots) \\ &= (a_2 + \alpha b_2, a_3 + \alpha b_3, \dots) \\ &= (a_2, a_3, \dots) + \alpha(b_2, b_3, \dots) \\ &= \lambda(\mathbf{a}) + \alpha\lambda(\mathbf{b}); \\ \rho(\mathbf{a} + \alpha\mathbf{b}) &= \rho(a_1 + \alpha b_1, a_2 + \alpha b_2, a_3 + \alpha b_3, \dots) \\ &= (0, a_1 + \alpha b_1, a_2 + \alpha b_2, \dots) \\ &= (0, a_1, a_2, \dots) + \alpha(0, b_1, b_2, \dots) \\ &= \rho(\mathbf{a}) + \alpha\rho(\mathbf{b}).\end{aligned}$$

So λ and ρ are linear transformations. $\square$

- (ii) Show that λ is onto, but not one-to-one.

Proof. To show that λ is onto, let $(a_1, a_2, \dots) \in \mathbf{V}$. Then

$$(a_1, a_2, \dots) = \lambda(0, a_1, a_2, \dots) \in \mathbf{R}(\lambda).$$

To show that λ is not one-to-one, we take $\mathbf{a} = (1, 0, 0, \dots)$; then $\mathbf{a} \neq \mathbf{0}$, but

$$\lambda(\mathbf{a}) = (0, 0, \dots) = \mathbf{0}.$$

Since $\mathbf{N}(\lambda) \neq \{\mathbf{0}\}$, it follows that λ is not one-to-one. $\square$

- (iii) Show that ρ is one-to-one, but not onto.

Proof. To show that ρ is one-to-one, assume that $\mathbf{a} = (a_1, a_2, \dots) \in \mathbf{N}(\rho)$. Then

$$\mathbf{0} = (0, 0, 0, \dots) = \rho(\mathbf{a}) = (0, a_1, a_2, \dots),$$

so $a_1 = 0, a_2 = 0, \dots$, showing that $\mathbf{a} = \mathbf{0}$.

To verify that ρ is not onto, we note that every vector in $\mathbf{R}(\rho)$ has first coordinate equal to 0, so $(1, 0, 0, \dots)$, which lies in $\mathbf{V}$, is not in the range. Thus, $\mathbf{R}(\rho) \neq \mathbf{V}$, proving that ρ is not onto. $\square$

- (iv) Show that $\lambda \circ \rho$ is the identity linear transformation $(\lambda \circ \rho)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbf{V}$; but that $\rho \circ \lambda$ is not the identity linear transformation.

Proof. Let $\mathbf{a} = (a_1, a_2, \dots)$ be a vector in $\mathbf{V}$. Then

$$\lambda \circ \rho(\mathbf{a}) = \lambda(\rho(a_1, a_2, \dots)) = \lambda(0, a_1, a_2, \dots) = (a_1, a_2, \dots) = \mathbf{a}.$$

Thus, $\lambda \circ \rho$ is the identity function.

To show that $\rho \circ \lambda$ is not the identity, note that

$$\rho \circ \lambda(1, 0, 0, 0, \dots) = \rho(0, 0, 0, \dots) = (0, 0, 0, \dots) \neq (1, 0, 0, 0, \dots).$$

(More generally, a vector $\mathbf{a} = (a_1, a_2, \dots)$ lies in $\mathbf{R}(\rho \circ \lambda)$ if and only if $a_1 = 0$.) $\square$

4. Let $T: \mathbf{P}_3(\mathbb{R}) \rightarrow \mathbf{M}_{2 \times 2}(\mathbb{R})$ be given by

$$T(p(x)) = \begin{pmatrix} p(1) & p'(0) \\ p(2) & p(0) \end{pmatrix}.$$

- (i) Prove that T is a linear transformation.

Proof. Note that for any value a and any two polynomials p and q , $(p+q)(a) = p(a) + q(a)$; and $(\alpha p)(a) = \alpha p(a)$; and since $(p+q)' = p' + q'$, it also holds that $(p+q)'(a) = p'(a) + q'(a)$. So let p and q be two vectors in $\mathbf{P}_3(\mathbb{R})$ and α a scalar. Then

$$\begin{aligned}(p + \alpha q)(1) &= p(1) + \alpha q(1), \\ (p + \alpha q)(2) &= p(2) + \alpha q(2), \\ (p + \alpha q)(0) &= p(0) + \alpha q(0), \\ (p + \alpha q)'(0) &= p'(0) + \alpha q'(0).\end{aligned}$$

So we have:

$$\begin{aligned}T(p + \alpha q) &= \begin{pmatrix} (p + \alpha q)(1) & (p + \alpha q)'(0) \\ (p + \alpha q)(2) & (p + \alpha q)(0) \end{pmatrix} = \begin{pmatrix} p(1) + \alpha q(1) & p'(0) + \alpha q'(0) \\ p(2) + \alpha q(2) & p(0) + \alpha q(0) \end{pmatrix} \\ &= \begin{pmatrix} p(1) & p'(0) \\ p(2) & p(0) \end{pmatrix} + \alpha \begin{pmatrix} q(1) & q'(0) \\ q(2) & q(0) \end{pmatrix} = T(p) + \alpha T(q),\end{aligned}$$

proving that T is a linear transformation. $\square$

- (ii) Find $\mathbf{N}(T)$ and $\mathbf{R}(T)$.

Answer. The nullspace is the collection of all $p \in \mathbf{P}_3(\mathbb{R})$ that map to the zero matrix; so that is all $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ for which $p(1) = p(2) = p(0) = p'(0) = 0$. We have $p'(x) = a_1 + 2a_2x + 3a_3x^2$; so this condition means

$$\begin{aligned}0 &= p(0) = a_0, \\ 0 &= p(1) = a_0 + a_1 + a_2 + a_3 \\ 0 &= p(2) = a_0 + 2a_1 + 4a_2 + 8a_3 \\ 0 &= p'(0) = a_1.\end{aligned}$$

The first and last tell us that $a_0 = a_1 = 0$. That means that we have $a_2 + a_3 = 0$, or equivalently $a_3 = -a_2$; and substituting into the third equation that gives $0 = 4a_2 - 8a_2$. So $a_2 = a_3 = 0$ as well.

Thus, $\mathbf{N}(T) = \{0\}$. Note this means that $\text{nullity}(T) = 0$.

By the Dimension Theorem, we know that

$$\dim(\mathbf{P}_3(\mathbb{R})) = 4 = \text{nullity}(T) + \text{rank}(T) = \text{rank}(T).$$

So $\text{rank}(T) = 4$. Since $\dim(\mathbf{M}_{2 \times 2}(\mathbb{R})) = 4$, and $\mathbf{R}(T)$ is a 4-dimensional subspace of $\mathbf{M}_{2 \times 2}(\mathbb{R})$, it follows that $\mathbf{R}(T) = \mathbf{M}_{2 \times 2}(\mathbb{R})$. $\square$

5. Let $T: \mathbf{V} \rightarrow \mathbf{W}$ be a linear transformation. Prove that the following two statements are equivalent:

- (i) T is one-to-one.
- (ii) If $S \subseteq \mathbf{V}$ is linearly independent in $\mathbf{V}$, then $T(S) = \{T(s) \mid s \in S\}$ is linearly independent in $\mathbf{W}$.

That is, T is one to one if and only if it sends linearly independent sets to linearly independent sets.

Proof. (1) $\implies$ (ii) Suppose that T is one to one, and that S is a linearly independent subset of $\mathbf{V}$. We want to prove that $T(S)$ is linearly independent. To that end, let $T(s_1), \dots, T(s_n) \in T(S)$ be pairwise distinct vectors, and $a_1, \dots, a_n$ be scalars such that

$$\mathbf{0}_W = a_1T(s_1) + \dots + a_nT(s_n) = T(a_1s_1 + \dots + a_ns_n).$$

Since T is one-to-one, this means that $a_1 s_1 + \cdots + a_n s_n \in \mathbf{N}(T) = \{\mathbf{0}_V\}$, so

$$a_1 s_1 + \cdots + a_n s_n = \mathbf{0}_V.$$

But $s_1, \dots, s_n$ are pairwise distinct elements of the linearly independent set S , so this equality implies that $a_1 = \cdots = a_n = 0$. Therefore, $T(S)$ is linearly independent, as claimed.

(ii) $\implies$ (i) Suppose that T sends linearly independent sets to linearly independent sets. If $\mathbf{v} \neq \mathbf{0}_V$, then $\{\mathbf{v}\}$ is linearly independent, and therefore $\{T(\mathbf{v})\} = T(\{\mathbf{v}\})$ is linearly independent; this implies that $T(\mathbf{v}) \neq \mathbf{0}_W$. By contrapositive, this means that if $T(\mathbf{v}) = \mathbf{0}_W$, then $\mathbf{v} = \mathbf{0}_V$; thus, T is one-to-one. $\square$

6. Let $T: V \rightarrow W$ be a linear transformation. Prove that the following two statements are equivalent:

- (i) T is surjective.
- (ii) If $S \subseteq V$ generates V , then $T(S)$ generates W .

That is, T is surjective if and only if it sends generating sets to generating sets.

Proof. Let us first note that for any subset S of V , we have

$$\text{span}(T(S)) = T(\text{span}(S)).$$

Indeed: since $S \subseteq \text{span}(S)$, then $T(S) \subseteq T(\text{span}(S))$. Since the image of a subspace of V is a subspace of W , we have that $T(\text{span}(S))$ is a subspace of W that contains $T(S)$, and therefore we must have that $\text{span}(T(S)) \subseteq T(\text{span}(S))$.

Conversely, given $\mathbf{v} \in \text{span}(S)$, we can express $\mathbf{v}$ as $\mathbf{v} = a_1 \mathbf{s}_1 + \cdots + a_n \mathbf{s}_n$ for some $\mathbf{s}_i \in S$. Then

$$T(\mathbf{v}) = T(a_1 \mathbf{s}_1 + \cdots + a_n \mathbf{s}_n) = a_1 T(\mathbf{s}_1) + \cdots + a_n T(\mathbf{s}_n) \in \text{span}(T(S)).$$

Therefore, $T(\text{span}(S)) \subseteq \text{span}(T(S))$, which proves the equality.

(i) $\implies$ (ii) Suppose that S is a generating set for V . Since T is surjective, we have that $T(V) = W$. So $W = T(V) = T(\text{span}(S)) = \text{span}(T(S))$, and we conclude that $T(S)$ generates W , as desired.

(ii) $\implies$ (i) Suppose that T maps generating sets to generating sets. Since V generates V , then $T(V)$ must be a generating set for W . Therefore, $\text{span}(T(V)) = W$. But $T(V)$ is already a subspace, so $\text{span}(T(V)) = T(V)$. Therefore, $T(V) = W$, so T is surjective, as claimed. $\square$