

Math 561 - Homework 4

SOLUTIONS

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- Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces, not necessarily finite dimensional. Prove that a linear transformation $T: \mathbf{V} \rightarrow \mathbf{W}$ is invertible if and only if for every subset S of $\mathbf{V}$, if S is a basis for $\mathbf{V}$, then $T(S)$ is a basis for $\mathbf{W}$.

Proof. $\implies$) Suppose that T is invertible, and S is a basis for $\mathbf{V}$. Since T is invertible, it is both one-to-one and onto. Since S is a basis, it is both linearly independent and a spanning set for $\mathbf{V}$.

We proved in Homework 3 that if T is one to one, then it sends linearly independent sets to linearly independent sets; and if T is surjective, then it sends spanning sets to spanning sets. So in this setting, we obtain that $T(S)$ is both linearly independent and a spanning set for $\mathbf{W}$; but that means that $T(S)$ is a basis.

Thus, if T is invertible, then it sends bases of $\mathbf{V}$ to bases for $\mathbf{W}$.

$\impliedby$) To prove that T is one-to-one, let L be a linearly independent subset of $\mathbf{V}$; we can extend L to a basis B , and we have that $T(B)$ is a basis for $\mathbf{W}$. In particular, $T(B)$ is linearly independent. Since $T(L) \subseteq T(B)$, and a subset of a linearly independent set is linearly independent, it follows that $T(L)$ is linearly independent. Therefore, T sends linearly independent sets to linearly independent sets, hence T is one-to-one.

To show that T is surjective, let G be a spanning set for $\mathbf{V}$; then G contains a basis B , and $T(B)$ is a basis for $\mathbf{W}$. Since $T(B) \subseteq T(G)$, and $\mathbf{W} = \text{span}(T(B)) \subseteq \text{span}(T(G)) \subseteq \mathbf{W}$, we have that $T(G)$ spans $\mathbf{W}$. Thus, T sends spanning sets to spanning sets, hence T is surjective.

Thus, T is invertible, as claimed. $\square$

REMARK: In fact, $T: \mathbf{V} \rightarrow \mathbf{W}$ is invertible if and only if there exists a basis S for $\mathbf{V}$ such that T is one-to-one when restricted to S , and $T(S)$ is a basis for $\mathbf{W}$.

- Let β be the following basis for $\mathbb{R}^4$:

$$\beta = \left\{ (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1) \right\}.$$

You may take for granted that β is a basis. Let T be the (unique) linear transformation $T: \mathbb{R}^4 \rightarrow \mathbf{P}_2(\mathbb{R})$, such that $T(1, 0, 0, 0) = 1 - x$, $T(1, 1, 0, 0) = 2x - x^2$, $T(1, 1, 1, 0) = \frac{1}{2}x + \frac{1}{3}x^2$, and $T(1, 1, 1, 1) = 0$.

Give a formula for $T(a, b, c, d)$ in terms of a , b , c , and d .

Answer. Let $(a, b, c, d) \in \mathbb{R}^4$. First we express this vector in terms of the basis β ; looking at the coordinates from last to first, it is straightforward to see that

$$(a, b, c, d) = (a - b)(1, 0, 0, 0) + (b - c)(1, 1, 0, 0) + (c - d)(1, 1, 1, 0) + d(1, 1, 1, 1).$$

Therefore, we have:

$$\begin{aligned} T(a, b, c, d) &= (a - b)T(1, 0, 0, 0) + (b - c)T(1, 1, 0, 0) + (c - d)T(1, 1, 1, 0) + dT(1, 1, 1, 1) \\ &= (a - b)(1 - x) + (b - c)(2x - x^2) + (c - d)\left(\frac{1}{2}x + \frac{1}{3}x^2\right) + d(0) \\ &= (a - b) + \left(-a + b + 2b - 2c + \frac{c - d}{2}\right)x + \left(-b + c + \frac{c - d}{3}\right)x^2 \\ &= (a - b) + \left(3b - a - \frac{3}{2}c - \frac{1}{2}d\right)x + \left(\frac{4}{3}c - b - \frac{1}{3}d\right)x^2. \end{aligned}$$

3. Let $T: \mathbf{P}_3(\mathbb{R}) \rightarrow \mathbf{M}_{2 \times 2}(\mathbb{R})$ be given by

$$T(p(x)) = \begin{pmatrix} p(1) & p'(0) \\ p(2) & p(0) \end{pmatrix}.$$

We proved in Homework 3 that T is a linear transformation. Let $\beta = [1, x, x^2, x^3]$ be the standard ordered basis for $\mathbf{P}_3(\mathbb{R})$, and let $\gamma = [E_{11}, E_{12}, E_{21}, E_{22}]$ be an ordered basis for $\mathbf{M}_{2 \times 2}(\mathbb{R})$.

(i) Find $[T]_{\beta}^{\gamma}$.

Answer. We calculate the value of T at each element of β , and write it in terms of the vectors in γ :

$$T(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1E_{11} + 0E_{12} + 1E_{21} + 1E_{22}$$

$$T(x) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = 1E_{11} + 1E_{12} + 2E_{21} + 0E_{22}$$

$$T(x^2) = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix} = 1E_{11} + 0E_{12} + 4E_{21} + 0E_{22}$$

$$T(x^3) = \begin{pmatrix} 1 & 0 \\ 8 & 0 \end{pmatrix} = 1E_{11} + 0E_{12} + 8E_{21} + 0E_{22}.$$

From this, we get

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) Use (i) to find $T(2 - \frac{3}{2}x + 5x^2 - x^3)$.

Answer. We have that

$$\left[2 - \frac{3}{2}x + 5x^2 - x^3\right]_{\beta} = \begin{pmatrix} 2 \\ -\frac{3}{2} \\ 5 \\ -1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} [T(2 - \frac{3}{2}x + 5x^2 - x^3)]_{\gamma} &= [T]_{\beta}^{\gamma} [2 - \frac{3}{2}x + 5x^2 - x^3]_{\beta} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -\frac{3}{2} \\ 5 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 - \frac{3}{2} + 5 - 1 \\ 0 - \frac{3}{2} + 0 + 0 \\ 2 - 3 + 20 - 8 \\ 2 + 0 + 0 + 0 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ -\frac{3}{2} \\ 11 \\ 2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$T(2 - \frac{3}{2}x + 5x^2 - x^3) = \frac{9}{2}E_{11} - \frac{3}{2}E_{12} + 11E_{21} + 2E_{22} = \begin{pmatrix} \frac{9}{2} & -\frac{3}{2} \\ 11 & 2 \end{pmatrix}.$$

4. Let $a \in \mathbb{R}$ be a real number, and define $T_a: \mathbf{P}_2(\mathbb{R}) \rightarrow \mathbf{P}_2(\mathbb{R})$ to be given by $T_a(p(x)) = p(x - a)$. Let β be the standard ordered basis for $\mathbf{P}_2(\mathbb{R})$.

- (i) Find $[T_a]_\beta^\beta$.

Answer. Recall that $\beta = [1, x, x^2]$. So

$$\begin{aligned} T_a(1) &= 1 \\ T_a(x) &= x - a \\ T_a(x^2) &= (x - a)^2 = x^2 - 2ax + a^2. \end{aligned}$$

So

$$[T]_\beta^\beta = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}.$$

- (ii) Find the value of c for which $T_a \circ T_b = T_c$.

Answer. We have that

$$T_a \circ T_b(p(x)) = T_a(p(x - b)) = p((x - b) - a) = p(x - b - a) = p(x - (a + b)).$$

So $c = a + b$.

- (iii) Use (ii) to find $[T_a]_\beta^\beta [T_b]_\beta^\beta$ without computing the matrix product explicitly.

Answer. We have:

$$[T_a]_\beta^\beta [T_b]_\beta^\beta = [T_a \circ T_b]_\beta^\beta = [T_{a+b}]_\beta^\beta = \begin{pmatrix} 1 & -(a+b) & (a+b)^2 \\ 0 & 1 & -2(a+b) \\ 0 & 0 & 1 \end{pmatrix}.$$

- (iv) Find $([T_a]_\beta^\beta)^{-1}$ without computing the matrix inverse.

Answer. Note that $(T_a)^{-1} = T_{-a}$, since T_0 is the identity, and $T_a \circ T_{-a} = T_{-a} \circ T_a = T_0$. So

$$([T_a]_\beta^\beta)^{-1} = [(T_a)^{-1}]_\beta^\beta = [T_{-a}]_\beta^\beta = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}.$$

5. Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces, and suppose that $T: \mathbf{V} \rightarrow \mathbf{W}$ and $U: \mathbf{W} \rightarrow \mathbf{V}$ are linear transformations.

- (i) Prove that if $\dim(\mathbf{V}) = \dim(\mathbf{W})$ and $UT = I_{\mathbf{V}}$, the identity on $\mathbf{V}$, then $U = T^{-1}$.

Proof. If $UT = I_{\mathbf{V}}$, then T is one-to-one (if a composition is one-to-one, then the first function applied must be one-to-one). So $T: \mathbf{V} \rightarrow \mathbf{W}$ is a one-to-one linear transformation between finite dimensional spaces of the same dimension; this implies that T is bijective, and therefore invertible.

Since T is invertible, T^{-1} exists. We then have

$$U = U \circ I_{\mathbf{W}} = U \circ (T \circ T^{-1}) = (U \circ T) \circ T^{-1} = I_{\mathbf{V}} \circ T^{-1} = T^{-1},$$

as desired. $\square$

- (ii) Give an example to show that (i) need not hold if we do not assume $\dim(\mathbf{V}) = \dim(\mathbf{W})$.

Answer. Let $T: \mathbb{R} \rightarrow \mathbb{R}^2$ be given by $T(a) = (a, 0)$ and $U: \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $U(a, b) = a$. Then $U \circ T(a) = U(a, 0) = a$, so $UT = I_{\mathbb{R}}$. But T is not surjective, so T is not invertible. Note that here $\dim(\mathbf{V}) = 1$ and $\dim(\mathbf{W}) = 2$.

- (iii) Give an example to show that (i) need not hold if $\dim(\mathbf{V}) = \dim(\mathbf{W})$ but we do not assume that the dimensions are finite.

Answer. Let $\mathbf{V} = \mathbf{W} = \mathbb{R}^\infty$ as in Problem 3 of Homework 3 (the vector space of real sequences), and let $\rho, \lambda: \mathbf{V} \rightarrow \mathbf{V}$ be the “right shift” and “left shift” operators, respectively. We proved in Homework 3 that $\lambda \circ \rho$ is the identity linear transformation, but that $\rho \circ \lambda$ is **not** the identity linear transformation; so letting $T = \rho$ and $U = \lambda$ gives us an example.

6. Show that if A and B are $n \times n$ matrices and $AB = I_n$, then $A = B^{-1}$.

Proof. Let $\mathbf{V} = \mathbf{W} = \mathbb{F}^n$, $T = L_B$, and $U = L_A$. Then we have that $L_A L_B = L_{AB} = L_{I_n} = I_{\mathbf{V}}$, so by Problem 5(i), we conclude that $L_A = (L_B)^{-1} = L_{B^{-1}}$. Therefore, $A = B^{-1}$.

7. Give an example of matrices A and B such that $AB = I_2$ the 2×2 identity matrix, but $A \neq B^{-1}$.

Answer. Think about linear transformations $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $U: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $UT = I_{\mathbb{R}^2}$. Then we can take A and B to be the coordinate matrices of U and T with respect to the standard bases, respectively.

For example, let $T(a, b) = (a, b, 0)$ and $U(x, y, z) = (x, y)$. Then

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

will work.

8. Let $\beta = [(1, 1, 1), (1, 0, 1), (1, 1, 2)]$; you may take for granted that β is a basis for $\mathbb{R}^3$. Let A be the matrix

$$A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}.$$

Find $[L_A]_{\beta}^{\beta}$.

Answer. METHOD 1: DIRECT COMPUTATION.

The linear transformation L_A sends the vector $\mathbf{x}$ to the vector $A\mathbf{x}$. To find $[L_A]_{\beta}^{\beta}$, we must find the values of L_A at each vector in β , and then express the answer in terms of β . Then we use that expression to find the coordinate vector $[L_A(\mathbf{x})]_{\beta}$, and these will form the columns of $[L_A]_{\beta}^{\beta}$. We have:

$$L_A(1, 1, 1) = A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix}$$

$$L_A(1, 0, 1) = A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 17 \\ 5 \\ 14 \end{pmatrix}$$

$$L_A(1, 1, 2) = A \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 22 \\ 22 \\ 28 \end{pmatrix}.$$

The first one is easy to express:

$$(18, 18, 18) = 18(1, 1, 1) + 0(1, 0, 1) + 0(1, 1, 2).$$

For the other two, we can solve the corresponding systems of linear equations, e.g., via row-reduction of an augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & | & 17 & 22 \\ 1 & 0 & 1 & | & 5 & 22 \\ 1 & 1 & 2 & | & 14 & 28 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & | & 17 & 22 \\ 0 & -1 & 0 & | & -12 & 0 \\ 0 & 0 & 1 & | & -3 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & | & 5 & 22 \\ 0 & 1 & 0 & | & 12 & 0 \\ 0 & 0 & 1 & | & -3 & 6 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 8 & 16 \\ 0 & 1 & 0 & | & 12 & 0 \\ 0 & 0 & 1 & | & -3 & 6 \end{pmatrix}.$$

Which means that

$$\begin{aligned} (17, 5, 14) &= 8(1, 1, 1) + 12(1, 0, 1) - 3(1, 1, 2) \\ (22, 22, 28) &= 16(1, 1, 1) + 0(1, 0, 1) + 6(1, 1, 2). \end{aligned}$$

Thus, we have

$$[L_A]_{\beta}^{\beta} = \begin{pmatrix} 18 & 8 & 16 \\ 0 & 12 & 0 \\ 0 & -3 & 6 \end{pmatrix}.$$

METHOD 2: USING THEOREM FROM CLASS.

In class, we said that if $A \in \mathbf{M}n \times n(\mathbb{F})$, and β is an ordered basis for $\mathbb{F}^n$, then $[L_A]_{\beta}^{\beta} = Q^{-1}AQ$, where Q is the matrix whose columns are the vectors in β ; that is, $Q = [I_{\mathbb{F}^n}]_{\beta}^{\text{st}}$, where st is the standard basis for $\mathbb{F}^n$.

Letting Q be the matrix

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

we can compute the inverse of Q by expressing the standard basis in terms of β :

$$\begin{aligned} (1, 0, 0) &= 1(1, 1, 1) + 1(1, 0, 1) - 1(1, 1, 2) \\ (0, 1, 0) &= 1(1, 1, 1) - 1(1, 0, 0) + 0(1, 1, 2) \\ (0, 0, 1) &= -1(1, 1, 1) + 0(1, 0, 1) + 1(1, 1, 2) \end{aligned}$$

so

$$Q^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

so we have

$$\begin{aligned} [L_A]_{\beta}^{\beta} &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 10 & -2 \\ 12 & -12 & 0 \\ -9 & 3 & 6 \end{pmatrix} \begin{pmatrix} 11 & 1 & \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 18 & 8 & 16 \\ 0 & 12 & 0 \\ 0 & -3 & 6 \end{pmatrix}, \end{aligned}$$

same answer as with Method 1.