

# Math 561 - Homework 5

SOLUTIONS

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1. Show that a function  $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$  is a linear transformation if and only if there exist functionals  $f_1, \dots, f_m \in (\mathbb{F}^n)^*$  such that

$$T(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$$

for all  $\mathbf{x} \in \mathbb{F}^n$ .

**Proof.** First, suppose  $f_1, \dots, f_m$  are linear functionals on  $\mathbb{F}^n$ , and  $T$  is defined as given. Since  $f_i(\mathbf{x}) \in \mathbb{F}$ , the expression makes sense. And because each  $f_i$  is linear, we have:

$$\begin{aligned} T(\mathbf{x} + \alpha\mathbf{y}) &= (f_1(\mathbf{x} + \alpha\mathbf{y}), \dots, f_m(\mathbf{x} + \alpha\mathbf{y})) \\ &= (f_1(\mathbf{x}) + \alpha f_1(\mathbf{y}), \dots, f_m(\mathbf{x}) + \alpha f_m(\mathbf{y})) \\ &= (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) + \alpha (f_1(\mathbf{y}), \dots, f_m(\mathbf{y})) \\ &= T(\mathbf{x}) + \alpha T(\mathbf{y}). \end{aligned}$$

Thus,  $T$  is a linear transformation.

Conversely, assume  $T$  is a linear transformation. Letting  $[g_1, \dots, g_m]$  be the dual basis of the standard ordered basis, we know that  $g_i(a_1, \dots, a_m) = a_i$ . Thus, if we let  $f_i = g_i T$ , then  $f_i$  is linear (composition of linear transformations), has domain  $\mathbb{F}^n$  and range  $\mathbb{F}$ , hence  $f_i \in (\mathbb{F}^n)^*$ ; and if  $T(\mathbf{x}) = (a_1, \dots, a_m)$ , then

$$f_i(\mathbf{x}) = g_i(T(\mathbf{x})) = g_i(a_1, \dots, a_m) = a_i.$$

Thus,  $T(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$ , as desired.  $\square$

2. Let  $\mathbf{V} = \mathbf{P}_n(\mathbb{F})$  and let  $c_0, c_1, \dots, c_n$  be distinct scalars in  $\mathbb{F}$ .

- (i) For  $0 \leq i \leq n$ , define  $f_i \in \mathbf{V}^*$  by  $f_i(p) = p(c_i)$ . Prove that  $[f_0, \dots, f_n]$  is a basis for  $\mathbf{V}^*$ .  
HINT: Apply any linear combination of these functionals that is equal to the zero function to the polynomial  $p(x) = (x - c_1) \cdots (x - c_n)$  to deduce the coefficient of  $f_0$  in the linear combination is zero.

**Proof.** Following the hint, assume that  $g = \alpha_0 f_0 + \alpha_1 f_1 + \cdots + \alpha_n f_n = 0$ . Since  $p(c_i) = 0$  for  $i = 1, \dots, n$ , we obtain  $0 = g(p(x)) = \alpha_0 p(c_0)$ . But  $p(c_0) \neq 0$ , since the  $c_i$  are distinct, so we conclude that  $\alpha_0 = 0$ .

Now replace  $p(x)$  with  $\frac{p(x)}{x - c_1}$ , which is zero precisely at  $c_2, \dots, c_n$ . Plugging into  $g$  we obtain  $0 = \alpha_1 p(c_1)$ , so  $\alpha_1 = 0$ . Continuing this way, we obtain  $\alpha_2 = \cdots = \alpha_n = 0$ , as desired.

Since  $\dim(\mathbf{V}^*) = \dim(\mathbf{V}) = n + 1$ , and  $f_0, \dots, f_n$  are linearly independent, it must be a basis for  $\mathbf{V}^*$ .  $\square$

- (ii) Conclude that there are unique polynomials  $p_0, \dots, p_n \in \mathbf{V}$  such that  $p_i(c_j) = \delta_{ij}$  (Kronecker's delta).

**Proof.** Consider  $\gamma = [v_0, \dots, v_n]$ , the dual basis of  $[f_0, \dots, f_n]$ ; where  $v_i(f_j) = \delta_{ij}$ . But this is a basis for  $\mathbf{V}^{**}$ , and we know that every element of  $\mathbf{V}^{**}$  is of the form  $\hat{\mathbf{x}}$  for some  $\mathbf{x} \in \mathbf{V}$ , where  $\hat{\mathbf{x}}(g) = g(\mathbf{x})$ . So  $\gamma$  has the form  $\gamma = [\hat{p}_0, \dots, \hat{p}_n]$  for some polynomials  $p_0, \dots, p_n \in \mathbf{V}$  with the property that

$$\delta_{ij} = \hat{p}_j(f_i) = f_i(p_j) = p_j(c_i).$$

So we have the desired polynomials.

To show that these polynomials are unique, note that we saw in class that for any vector space  $\mathbf{V}$  and vector  $\mathbf{v} \in \mathbf{V}$ , there exists  $\mathbf{g} \in \mathbf{V}^*$  such that  $\mathbf{g}(\mathbf{v}) \neq 0$  (namely, extend  $\mathbf{v}$  to a basis, and then define  $\mathbf{g}$  to be 1 at  $\mathbf{v}$  and 0 elsewhere). If  $p$  and  $q$  are two polynomials such that  $f_i(p) = f_i(q)$  for  $i = 0, \dots, n$ , then because  $f_0, \dots, f_n$  span  $\mathbf{V}^*$  it follows that  $\mathbf{g}(p - q) = 0$  for all  $\mathbf{g} \in \mathbf{V}^*$ ; and this implies that  $p - q = \mathbf{0}$ , hence  $p = q$ . This means that the polynomials  $p_i(x)$  are completely determined by the values on  $c_0, \dots, c_n$ , so they are unique.  $\square$

- (iii) For any scalars  $a_0, \dots, a_n$ , not necessarily distinct, deduce the existence of a unique polynomial  $q(x)$  of degree at most  $n$  such that  $q(c_i) = a_i$  for  $i = 0, 1, \dots, n$ . In fact,

$$q(x) = \sum_{i=0}^n a_i p_i(x).$$

**Proof.** As noted above, if we specify the value of  $f_i(q)$  for  $i = 0, \dots, n$ , then this uniquely determines the polynomial  $q$ , so there is at most one polynomial with this property. Thus, it suffices to show that  $q(x)$  has this property; and indeed, we have

$$q(c_j) = f_j \left( \sum_{i=0}^n a_i p_i(x) \right) = \sum_{i=0}^n a_i p_i(c_j) = \sum_{i=0}^n a_i \delta_{ij} = a_j. \quad \square$$

- (iv) Deduce the Lagrange Interpolation Formula: for all  $p(x) \in \mathbf{V}$ ,

$$p(x) = \sum_{i=0}^n p(c_i) p_i(x).$$

**Proof.** If  $p(x)$  has degree at most  $n$ , then it is completely determined by its values at  $c_0, \dots, c_n$ , (that is, the value of the elements of  $\gamma$  at  $p(x)$ , as noted above). The formula now follows from (iii).  $\square$

- (v) For each  $i$ ,  $i = 0, 1, \dots, n$ , define

$$f_i(x) = \frac{(x - c_0) \cdots (x - c_{i-1})(x - c_{i+1}) \cdots (x - c_n)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)} = \prod_{\substack{k=0 \\ k \neq i}}^n \frac{x - c_k}{c_i - c_k}.$$

Verify that  $f_i(c_j) = \delta_{ij}$ , and conclude that  $f_i = p_i$ , where  $p_i$  are the polynomials whose existence was established in (ii).

**Proof.** We have

$$f_i(c_j) = \prod_{\substack{k=0 \\ k \neq i}}^n \frac{c_j - c_k}{c_i - c_k} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, \end{cases}$$

with the last equality because if  $i \neq j$ , then the  $j$ th factor is  $\frac{c_j - c_j}{c_i - c_j} = 0$ ; and if  $i = j$ , then each factor is  $\frac{c_i - c_j}{c_i - c_j} = 1$ .

Thus,  $f_i(c_j) = \delta_{ij}$ , as desired; and the uniqueness from (iii) gives that  $f_i = p_i$ , as claimed.  $\square$

- (vi) Find the unique polynomial of degree at most 4 such that  $p(0) = 1$ ,  $p(1) = 2$ ,  $p(2) = 4$ ,  $p(3) = 8$ , and  $p(4) = 16$ .

**Answer.** First we calculate  $p_0, \dots, p_4$ :

$$\begin{aligned} p_0(x) &= \prod_{i=1}^4 \frac{x-i}{0-i} = \frac{1}{24} (x^4 - 10x^3 + 35x^2 - 50x + 24) \\ p_1(x) &= \left( \frac{x}{1-0} \right) \left( \prod_{i=2}^4 \frac{x-i}{1-i} \right) = -\frac{1}{6} (x^4 - 9x^3 + 26x^2 - 24x) \\ p_2(x) &= \frac{x(x-1)(x-3)(x-4)}{(2-0)(2-1)(2-3)(2-4)} = \frac{1}{4} (x^4 - 8x^3 + 19x^2 - 12x) \\ p_3(x) &= \frac{x(x-1)(x-2)(x-4)}{(3-0)(3-1)(3-2)(3-4)} = -\frac{1}{6} (x^4 - 7x^3 + 14x^2 - 8x) \\ p_4(x) &= \frac{x(x-1)(x-2)(x-3)}{(4-0)(4-1)(4-2)(4-3)} = \frac{1}{24} (x^4 - 6x^3 + 11x^2 - 6x). \end{aligned}$$

The polynomial we want is then

$$\begin{aligned} p(x) &= p_0(x) + 2p_1(x) + 4p_2(x) + 8p_3(x) + 16p_4(x) \\ &= \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{11}{24}x^2 + \frac{7}{12}x + 1. \end{aligned}$$

(vii) Find  $p(5)$ , the value of the polynomial  $p(x)$  you found in (vi) at  $x = 5$ .

**Answer.** We have

$$\begin{aligned} p(5) &= \frac{625}{24} - \frac{125}{12} + \frac{275}{24} + \frac{35}{12} + 1 \\ &= \frac{625 - 250 + 275 + 70 + 24}{24} = \frac{744}{24} = 31. \end{aligned}$$

This of course means that the next number in the sequence

$$1, \quad 2, \quad 4, \quad 8, \quad 16$$

is 31.  $\square$

3. Find the dimension of, and a basis for, the solution space of

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 0 \\ 3x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 0. \end{array}$$

**Answer.** A basis for the solution space is given by the dimension of the nullspace of  $L_A$ , where

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 3 & 2 & -1 & 3 \end{pmatrix}.$$

Subtracting three times the first row from the second row, then dividing the second row by  $-4$ , and subtracting twice the second row from the first row, we get

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & -4 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 \end{pmatrix}.$$

Here we can read off that the nullspace has dimension 2, since rank of this matrix is 2, and from the Rank-Nullity Theorem we know that the rank plus the nullity must equal the number of columns (here, four). Since this final matrix tells us the system is equivalent to the system

$$\begin{array}{ccccccc} x_1 & & & + & x_3 & + & x_4 & = & 0 \\ & x_2 & - & 2x_3 & & & & = & 0, \end{array}$$

so a basis for the nullspace is given by the vectors

$$\begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

4. Find all solutions to the system of linear equations

$$\begin{array}{rrrrrr} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 1 \\ 3x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 3. \end{array}$$

**Answer.** The solutions to this system will be given by a particular solution  $\mathbf{s}_0$ , plus an arbitrary solution to the corresponding homogeneous system. It is clear that  $x_1 = 1, x_2 = x_3 = x_4 = 0$  is a solution to this system, so the solutions to this system are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - s - t \\ 2s \\ s \\ t \end{pmatrix}, \quad s, t \in \mathbb{F}.$$

5. Determine whether the following system of linear equations has a solution. You do not need to find the solutions if they exist, just determine whether there is a solution or not.

$$\begin{array}{rrrrrr} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 3 \\ x_1 & - & 4x_2 & + & 7x_3 & = & 4. \end{array}$$

**Answer.** We row-reduce the augmented matrix of the system to see if there are solutions.

$$\begin{aligned} \left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & -4 & 7 & 4 \end{array} \right) & \xrightarrow{R_2 - 2R_1} \left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 1 & -4 & 7 & 4 \end{array} \right) \xrightarrow{R_3 - R_1} \left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & -6 & 8 & 3 \end{array} \right) \\ & \xrightarrow{R_3 - 2R_2} \left( \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right). \end{aligned}$$

So we see that the coefficient matrix has rank 2, while the augmented matrix has rank 3; so this system has no solutions.